Exercise 6: Some things about classification.

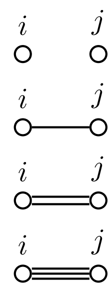
(1) Let E be a euclidean space/$\mathbb{R}$ with inner product $\langle , \rangle$ (of any big dimension). Call a finite subset $A = \{\alpha_1, \dots, \alpha_r\} \subset E$ *admissible* if

(i) A is a set of linearly independent unit vectors ($\langle \alpha_i, \alpha_i \rangle = 1$),

(ii) $\langle \alpha_i, \alpha_j \rangle \leq 0$ whenever $i \neq j$, and

(iii) $4\langle \alpha_i, \alpha_j \rangle^2 \in \{0, 1, 2, 3\}$ whenever $i \neq j$.

Associate to any admissible set A a graph $\Gamma(A)$ (called the Coxeter diagram) with vertices labeled by elements of A (or i short for α_i), with $m_{i,j} = 4\langle \alpha_i, \alpha_j \rangle^2$ edges connecting i to j:



if $4\langle \alpha_i, \alpha_j \rangle^2 = 0$,

if $4\langle \alpha_i, \alpha_j \rangle^2 = 1$,

if $4\langle \alpha_i, \alpha_j \rangle^2 = 2$,

if $4\langle \alpha_i, \alpha_j \rangle^2 = 3$.

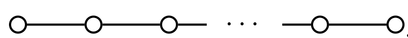
Let $A = \{\alpha_1, \dots, \alpha_r\}$ be an admissible set yielding a connected graph $\Gamma(A)$.

(a) Show that the number of pairs of vertices connected by at least one edge strictly less than r.
[What is the condition on vertices being adjacent? Consider $\langle \alpha, \alpha \rangle$ where $\alpha = \sum_A \alpha_i$.]

(b) Show that $\Gamma(A)$ contains no cycles. [Note that any subset of an admissible set is admissible.]

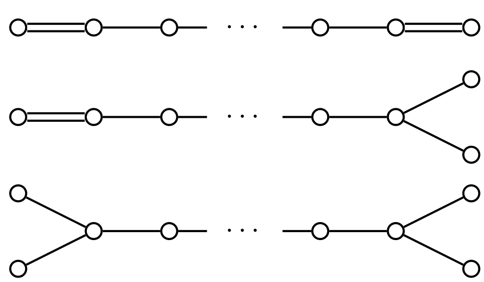
(c) Show that the degree (counting multiple edges) of any vertex in $\Gamma(A)$ is no more than three.
[Take a vertex $\alpha \in A$, and let S be the set containing α together with its neighborhood (the vertices adjacent to it). Note that in the span of S is a unit vector β which is orthogonal to $S - \{\alpha\}$, so that $\alpha = \sum_{\gamma \in S - \{\alpha\} + \{\beta\}} \langle \alpha, \gamma \rangle \gamma$ and $\langle \alpha, \beta \rangle \neq 0$ (why??).]

(d) Show that if $S \subseteq A$ has graph

$\Gamma(S) =$  ,

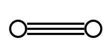
then $A' = A - S + \{\sum_S \alpha\}$ is admissible (with graph $\Gamma(A')$ obtained by collapsing the subgraph $\Gamma(S)$ to a single vertex).

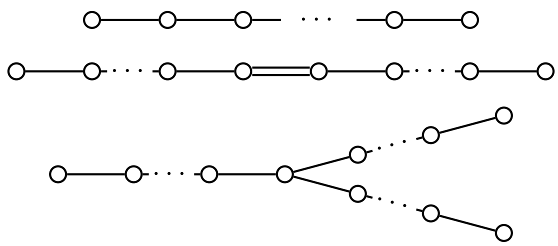
(e) Show that $\Gamma(A)$ cannot contain any of the following graphs as subgraphs:



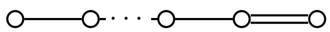
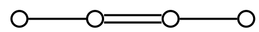
[Use the previous part]

(f) Show that the only remaining possible graphs associated to admissible sets are of one of the following four forms:



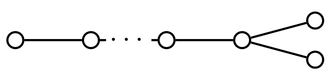
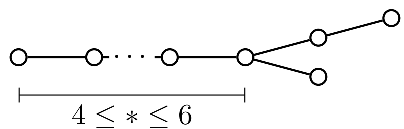


(g) Show the only possible graphs of the third type are

 and 

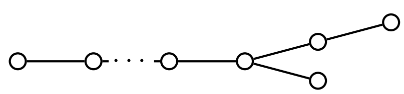
[Suppose the vectors corresponding to the vertices to the left of the double bond are $\lambda_1, \dots, \lambda_\ell$ (from left to right) and the vertices to the rights of the double bond are $\mu_1, \dots, \mu_m$ (from right to left). Let $\lambda = \sum_i i\lambda_i$ and $\mu = \sum_i i\mu_i$. Show that $\langle \lambda, \lambda \rangle = \ell(\ell+1)/2$, $\langle \mu, \mu \rangle = m(m+1)/2$, and $\langle \lambda, \mu \rangle^2 = \ell^2 m^2/2$, and use the Cauchy-Schwarz inequality for inner products.]

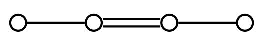
(h) Bonus: Show the only graphs of the fourth kind are

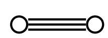
 and 

[This is like the previous part, only more so]

(2) Show that by normalizing the elements of any base B for a set of roots R, you get an admissible set A. From part 1, what's left over? Show that there's an admissible set associated to every remaining graph by displaying existence. Namely, associate most of the remaining possible graphs to a classical root systems (showing existence), and take for granted that the remaining five are associated to the *exceptional simple Lie algebras*, E_6, E_7, E_8, F_4, and G_2:

E_6, E_7, E_8 : 

F_4 : 

G_2 : 

(3) A *Dynkin diagram* associated to a base B for a root system is a decorated Coxeter graph for the associated normalized admissible set. If α_i is adjacent to α_j, and the root β_i associated to α_i is longer than the root β_j associated to α_j, decorate the $m_{i,j}$ edges connecting α_i to α_j with an arrow pointing to α_i (the normalization of the longer root).

Classify all (finite type) connected Dynkin diagrams.