11. Let Φ be irreducible. Prove that $\Phi^\vee$ is also irreducible. If Φ has all roots of equal length, so does $\Phi^\vee$ (and then $\Phi^\vee$ is isomorphic to Φ). On the other hand, if Φ has two root lengths, then so does $\Phi^\vee$; but if α is long, then $\alpha^\vee$ is short (and vice versa). Use this fact to prove that Φ has a unique maximal short root (relative to the partial order $\prec$ defined by Δ).
12. Let $\lambda \in \mathfrak{C}(\Delta)$. If $\sigma\lambda = \lambda$ for some $\sigma \in \mathscr{W}$, then $\sigma = 1$.
13. The only reflections in $\mathscr{W}$ are those of the form σ_α ($\alpha \in \Phi$). [A vector in the reflecting hyperplane would, if orthogonal to no root, be fixed only by the identity in $\mathscr{W}$.]
14. Prove that each point of E is $\mathscr{W}$-conjugate to a point in the closure of the fundamental Weyl chamber relative to a base Δ. [Enlarge the partial order on E by defining $\mu \prec \lambda$ iff $\lambda - \mu$ is a nonnegative **R**-linear combination of simple roots. If $\mu \in$ E, choose $\sigma \in \mathscr{W}$ for which $\lambda = \sigma\mu$ is maximal in this partial order.]

Notes

The exposition here is an expanded version of that in Serre [2].

11. Classification

In this section Φ denotes a root system of rank ℓ, $\mathscr{W}$ its Weyl group, Δ a base of Φ.

11.1. Cartan matrix of Φ

Fix an ordering $(\alpha_1, \ldots, \alpha_\ell)$ of the simple roots. The matrix $(\langle \alpha_i, \alpha_j \rangle)$ is then called the **Cartan matrix** of Φ. Its entries are called **Cartan integers.** *Examples*: For the systems of rank 2, the matrices are:

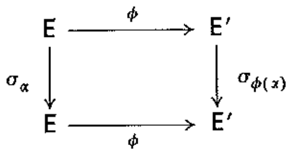
$$A_1 \times A_1 \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}; \quad A_2 \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}; \quad B_2 \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}; \quad G_2 \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}.$$

The matrix of course depends on the chosen ordering, but this is not very serious. The important point is that the Cartan matrix is independent of the choice of Δ, thanks to the fact (Theorem 10.3(b)) that $\mathscr{W}$ acts transitively on the collection of bases. The Cartan matrix is nonsingular, as in (8.5), since Δ is a basis of E. It turns out to characterize Φ completely.

Proposition. *Let $\Phi' \subset E'$ be another root system, with base $\Delta' = \{\alpha'_1, \ldots, \alpha'_\ell\}$. If $\langle \alpha_i, \alpha_j \rangle = \langle \alpha'_i, \alpha'_j \rangle$ for $1 \le i, j \le \ell$, then the bijection $\alpha_i \mapsto \alpha'_i$ extends (uniquely) to an isomorphism ϕ: $E \to E'$ mapping Φ onto Φ' and satisfying $\langle \phi(\alpha), \phi(\beta) \rangle = \langle \alpha, \beta \rangle$ for all $\alpha, \beta \in \Phi$. Therefore, the Cartan matrix of Φ determines Φ up to isomorphism.*

Proof. Since Δ (resp. Δ') is a basis of E (resp. E′), there is a unique vector space isomorphism ϕ: $E \to E'$ sending α_i to α'_i ($1 \le i \le \ell$). If $\alpha, \beta \in \Delta$, the hypothesis insures that $\sigma_{\phi(\alpha)}(\phi(\beta)) = \sigma_{\alpha'}(\beta') = \beta' - \langle \beta', \alpha' \rangle \alpha' = \phi(\beta) -$

$\langle\beta, \alpha\rangle\, \phi(\alpha) = \phi(\beta - \langle\beta, \alpha\rangle\, \alpha) = \phi(\sigma_\alpha(\beta))$. In other words, the following diagram commutes for each $\alpha \in \Delta$:

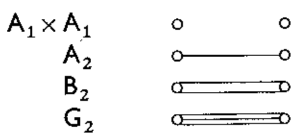


The respective Weyl groups $\mathscr{W}$, $\mathscr{W}'$ are generated by simple reflections (Theorem 10.3(d)), so it follows that the map $\sigma \mapsto \phi \circ \sigma \circ \phi^{-1}$ is an isomorphism of $\mathscr{W}$ onto $\mathscr{W}'$, sending σ_α to $\sigma_{\phi(\alpha)}$ ($\alpha \in \Delta$). But each $\beta \in \Phi$ is conjugate under $\mathscr{W}$ to a simple root (Theorem 10.3(c)), say $\beta = \sigma(\alpha)$ ($\alpha \in \Delta$). This in turn forces $\phi(\beta) = (\phi \circ \sigma \circ \phi^{-1})(\phi(\alpha)) \in \Phi'$. It follows that ϕ maps Φ onto Φ'; moreover, the formula for a reflection shows that ϕ preserves all Cartan integers. ▯

The proposition shows that it is possible in principle to recover Φ from a knowledge of the Cartan integers. In fact, it is not too hard to devise a practical algorithm for writing down all roots (or just all positive roots). Probably the best approach is to consider *root strings* (9.4). Start with the roots of height one, i.e., the simple roots. For any pair $\alpha_i \neq \alpha_j$, the integer r for the α_j-string through α_i is 0 (i.e., $\alpha_i - \alpha_j$ is not a root, thanks to Lemma 10.1), so the integer q equals $-\langle\alpha_i, \alpha_j\rangle$. This enables us in particular to write down all roots α of height 2, hence all integers $\langle\alpha, \alpha_j\rangle$. For each root α of height 2, the integer r for the α_j-string through α can be determined easily, since α_j can be subtracted at most once (why?), and then q is found, because we know $r - q = \langle\alpha, \alpha_j\rangle$. The corollary of Lemma 10.2A assures us that all positive roots are eventually obtained if we repeat this process enough times.

11.2. Coxeter graphs and Dynkin diagrams

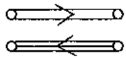
If α, β are distinct positive roots, then we know that $\langle\alpha, \beta\rangle\langle\beta, \alpha\rangle = 0$, 1, 2, or 3 (9.4). Define the **Coxeter graph** of Φ to be a graph having ℓ vertices, the ith joined to the jth ($i \neq j$) by $\langle\alpha_i, \alpha_j\rangle\langle\alpha_j, \alpha_i\rangle$ edges. *Examples*:

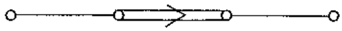


The Coxeter graph determines the numbers $\langle\alpha_i, \alpha_j\rangle$ in case all roots have equal length, since then $\langle\alpha_i, \alpha_j\rangle = \langle\alpha_j, \alpha_i\rangle$. In case more than one root length occurs (e.g., B_2 or G_2), the graph fails to tell us which of a pair of vertices should correspond to a short simple root, which to a long (in case these vertices are joined by two or three edges). (It can, however, be proved that the Coxeter graph determines the Weyl group completely, essentially because it determines the orders of products of generators of $\mathscr{W}$, cf. Exercise 9.3.)

Whenever a double or triple edge occurs in the Coxeter graph of Φ, we can add an arrow pointing to the shorter of the two roots. This additional information allows us to recover the Cartan integers; we call the resulting figure the **Dynkin diagram** of Φ. (As before, this depends on the numbering of simple roots.) For example:

B_2
G_2



Another example: Given the diagram  (which turns out to be associated with the root system F_4), the reader can easily recover the Cartan matrix

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}.$$

11.3. Irreducible components

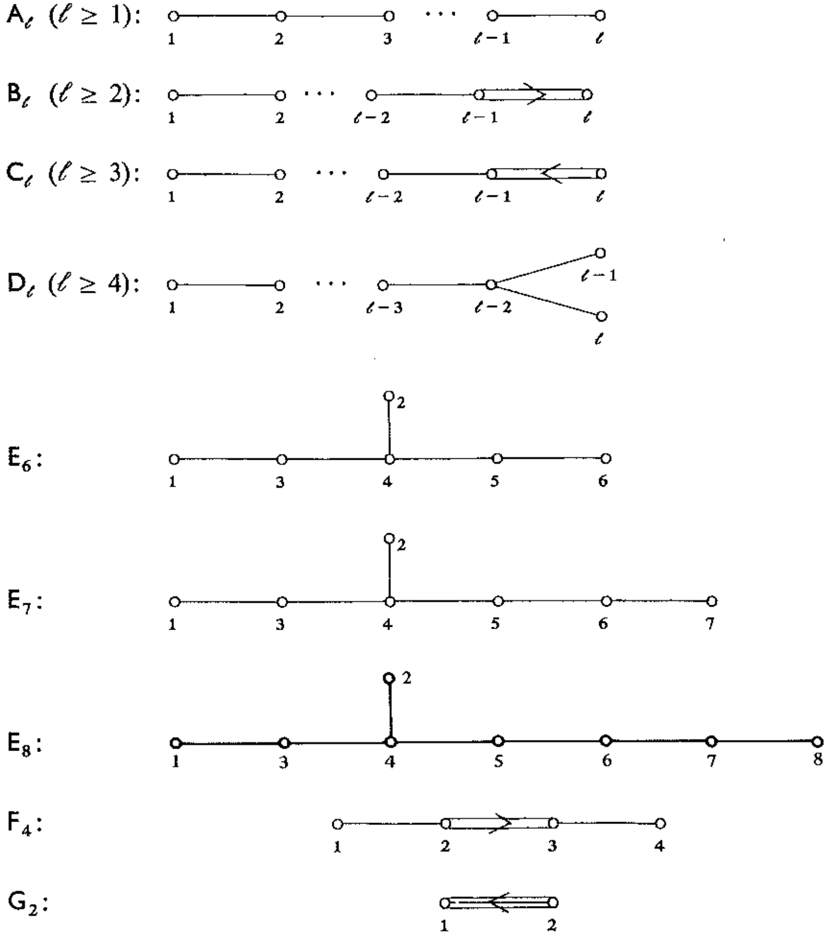
Recall (10.4) that Φ is irreducible if and only if Φ (or, equivalently, Δ) cannot be partitioned into two proper, orthogonal subsets. It is clear that Φ is *irreducible if and only if its Coxeter graph is connected* (in the usual sense). In general, there will be a number of connected components of the Coxeter graph; let $\Delta = \Delta_1 \cup \ldots \cup \Delta_t$ be the corresponding partition of Δ into mutually orthogonal subsets. If E_i is the span of Δ_i, it is clear that $\mathsf{E} = \mathsf{E}_1 \oplus \ldots \oplus \mathsf{E}_t$ (orthogonal direct sum). Moreover, the $\mathbf{Z}$-linear combinations of Δ_i which are roots (call this set Φ_i) obviously form a root system in E_i, whose Weyl group is the restriction to E_i of the subgroup of $\mathscr{W}$ generated by all σ_α $(\alpha \in \Delta_i)$. Finally, each E_i is $\mathscr{W}$-invariant (since $\alpha \notin \Delta_i$ implies that σ_α acts trivially on E_i), so the (easy) argument required for Exercise 9.1 shows immediately that each root lies in one of the E_i, i.e., $\Phi = \Phi_1 \cup \ldots \cup \Phi_t$.

Proposition. Φ *decomposes (uniquely) as the union of irreducible root systems* Φ_i *(in subspaces* E_i *of* E*) such that* $\mathsf{E} = \mathsf{E}_1 \oplus \ldots \oplus \mathsf{E}_t$ *(orthogonal direct sum).* ☐

11.4. Classification theorem

The discussion in (11.3) shows that it is sufficient to classify the irreducible root systems, or equivalently, the connected Dynkin diagrams (cf. Proposition 11.1).

Theorem. *If* Φ *is an irreducible root system of rank* ℓ, *its Dynkin diagram is one of the following (*ℓ *vertices in each case):*



The restrictions on ℓ for types $A_\ell - D_\ell$ are imposed in order to avoid duplication. Relative to the indicated numbering of simple roots, the corresponding Cartan matrices are given in Table 1. Inspection of the diagrams listed above reveals that in all cases except B_ℓ, C_ℓ, the Dynkin diagram can be deduced from the Coxeter graph. However, B_ℓ and C_ℓ both come from a single Coxeter graph, and differ in the relative numbers of short and long simple roots. (These root systems are actually dual to each other, cf. Exercise 5.)

Proof of Theorem. The idea of the proof is to classify first the possible Coxeter graphs (ignoring relative lengths of roots), then see what Dynkin diagrams result. Therefore, we shall merely apply some elementary euclidean geometry to finite sets of vectors whose pairwise angles are those prescribed by the Coxeter graph. Since we are ignoring lengths, it is easier to work for the time being with sets of unit vectors. For maximum flexibility, we make

Table 1. *Cartan matrices*

$$A_\ell:\quad \begin{pmatrix} 2 & -1 & 0 & & & \cdot & \cdot & \cdot & & 0 \\ -1 & 2 & -1 & 0 & & \cdot & \cdot & \cdot & & 0 \\ 0 & -1 & 2 & -1 & 0 & \cdot & \cdot & \cdot & & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & & \cdot & \cdot & \cdot & -1 & 2 \end{pmatrix}$$

$$B_\ell:\quad \begin{pmatrix} 2 & -1 & 0 & & \cdot & \cdot & \cdot & & & 0 \\ -1 & 2 & -1 & 0 & \cdot & \cdot & \cdot & & & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & & \cdot & \cdot & \cdot & -1 & 2 & -2 \\ 0 & 0 & 0 & & \cdot & \cdot & \cdot & 0 & -1 & 2 \end{pmatrix}$$

$$C_\ell:\quad \begin{pmatrix} 2 & -1 & 0 & & \cdot & \cdot & \cdot & & & 0 \\ -1 & 2 & -1 & & \cdot & \cdot & \cdot & & & 0 \\ 0 & -1 & 2 & -1 & \cdot & \cdot & \cdot & & & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & & \cdot & \cdot & \cdot & -1 & 2 & -1 \\ 0 & 0 & & & \cdot & \cdot & \cdot & 0 & -2 & 2 \end{pmatrix}$$

$$D_\ell:\quad \begin{pmatrix} 2 & -1 & 0 & & \cdot & \cdot & \cdot & & & 0 \\ -1 & 2 & -1 & & \cdot & \cdot & \cdot & & & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & & \cdot & \cdot & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & & \cdot & \cdot & & -1 & 2 & -1 & -1 \\ 0 & 0 & & \cdot & \cdot & & 0 & -1 & 2 & 0 \\ 0 & 0 & & \cdot & \cdot & & 0 & -1 & 0 & 2 \end{pmatrix}$$

$$E_6:\quad \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

$$E_7:\quad \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

$$E_8:\quad \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

$$F_4:\quad \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}. \qquad G_2:\quad \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

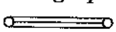
only the following assumptions: E is a euclidean space (of arbitrary dimension), $\mathfrak{A} = \{\varepsilon_1, \ldots, \varepsilon_n\}$ is a set of n linearly independent unit vectors which satisfy $(\varepsilon_i, \varepsilon_j) \leq 0$ $(i \neq j)$ and $4(\varepsilon_i, \varepsilon_j)^2 = 0, 1, 2,$ or 3 $(i \neq j)$. Such a set of vectors is called (for brevity) **admissible**. (*Example*: Elements of a base for a root system, each divided by its length.) We attach a graph Γ to the set $\mathfrak{A}$ just as we did above to the simple roots in a root system, with vertices i and j $(i \neq j)$ joined by $4(\varepsilon_i, \varepsilon_j)^2$ edges. Now our task is to determine all the connected graphs associated with admissible sets of vectors (these include all connected Coxeter graphs). This we do in steps, the first of which is obvious. (Γ is not assumed to be connected until later on.)

(1) *If some of the ε_i are discarded, the remaining ones still form an admissible set, whose graph is obtained from Γ by omitting the corresponding vertices and all incident edges.*

(2) *The number of pairs of vertices in Γ connected by at least one edge is strictly less than n.* Set $\varepsilon = \sum_{i=1}^{n} \varepsilon_i$. Since the ε_i are linearly independent, $\varepsilon \neq 0$. So $0 < (\varepsilon, \varepsilon) = n + 2\sum_{i<j} (\varepsilon_i, \varepsilon_j)$. Let i, j be a pair of (distinct) indices for which $(\varepsilon_i, \varepsilon_j) \neq 0$ (i.e., let vertices i and j be joined). Then $4(\varepsilon_i, \varepsilon_j)^2 = 1, 2,$ or 3, so in particular $2(\varepsilon_i, \varepsilon_j) \leq -1$. In view of the above inequality, the number of such pairs cannot exceed $n-1$.

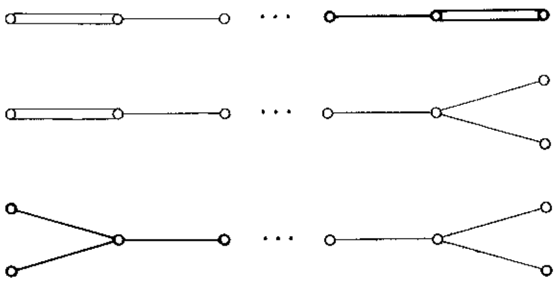
(3) *Γ contains no cycles.* A cycle would be the graph Γ' of an admissible subset $\mathfrak{A}'$ of $\mathfrak{A}$ (cf. (1)), and then Γ' would violate (2), with n replaced by Card $\mathfrak{A}'$.

(4) *No more than three edges can originate at a given vertex of Γ.* Say $\varepsilon \in \mathfrak{A}$, and $\eta_1, \ldots, \eta_k$ are the vectors in $\mathfrak{A}$ connected to ε (by 1, 2, or 3 edges each), i.e., $(\varepsilon, \eta_i) < 0$ with $\varepsilon, \eta_1, \ldots, \eta_k$ all distinct. In view of (3), no two η's can be connected, so $(\eta_i, \eta_j) = 0$ for $i \neq j$. Because $\mathfrak{A}$ is linearly independent, some unit vector η_0 in the span of $\varepsilon, \eta_1, \ldots, \eta_k$ is orthogonal to $\eta_1, \ldots, \eta_k$; clearly $(\varepsilon, \eta_0) \neq 0$ for such η_0. Now $\varepsilon = \sum_{i=0}^{k} (\varepsilon, \eta_i)\eta_i$, so $1 = (\varepsilon, \varepsilon) = \sum_{i=0}^{k} (\varepsilon, \eta_i)^2$. This forces $\sum_{i=1}^{k} (\varepsilon, \eta_i)^2 < 1$, or $\sum_{i=1}^{k} 4(\varepsilon, \eta_i)^2 < 4$. But $4(\varepsilon, \eta_i)^2$ is the number of edges joining ε to η_i in Γ.

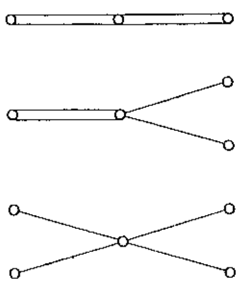
(5) *The only connected graph Γ of an admissible set $\mathfrak{A}$ which can contain a triple edge is*  *(the Coxeter graph G_2).* This follows at once from (4).

(6) *Let $\{\varepsilon_1, \ldots, \varepsilon_k\} \subset \mathfrak{A}$ have subgraph*  *(a simple chain in Γ. If $\mathfrak{A}' = (\mathfrak{A} - \{\varepsilon_1, \ldots, \varepsilon_k\}) \cup \{\varepsilon\}$, $\varepsilon = \sum_{i=1}^{k} \varepsilon_i$, then $\mathfrak{A}'$ is admissible.* (The graph of $\mathfrak{A}'$ is obtained from Γ by shrinking the simple chain to a point.) Linear independence of $\mathfrak{A}'$ is obvious. By hypothesis, $2(\varepsilon_i, \varepsilon_{i+1}) = -1$ $(1 \leq i \leq k-1)$, so $(\varepsilon, \varepsilon) = k + 2\sum_{i<j} (\varepsilon_i, \varepsilon_j) = k-(k-1) = 1$. So ε is a unit vector. Any $\eta \in \mathfrak{A} - \{\varepsilon_1, \ldots, \varepsilon_k\}$ can be connected to at most one of $\varepsilon_1, \ldots, \varepsilon_k$ (by (3)), so $(\eta, \varepsilon) = 0$ or else $(\eta, \varepsilon) = (\eta, \varepsilon_i)$ for $1 \leq i \leq k$. In either case, $4(\eta, \varepsilon)^2 = 0, 1, 2,$ or 3.

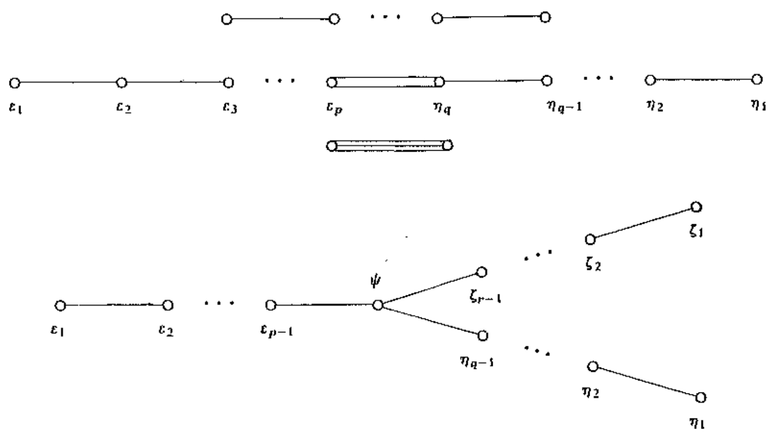
(7) Γ *contains no subgraph of the form*:

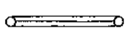


Suppose one of these graphs occurred in Γ; by (1) it would be the graph of an admissible set. But (6) allows us to replace the simple chain in each case by a single vertex, yielding (respectively) the following graphs which violate (4):

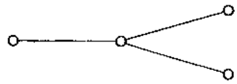


(8) *Any connected graph* Γ *of an admissible set has one of the following forms*:

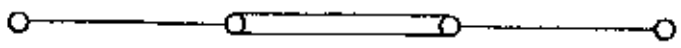
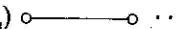
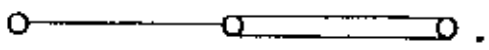


Indeed, only  contains a triple edge, by (5). A connected graph containing more than one double edge would contain a subgraph

which (7) forbids, so at most one double edge occurs. Moreover, if Γ has a double edge, it cannot also have a "node" (branch point)

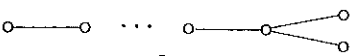


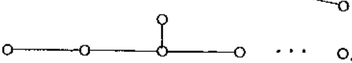
(again by (7)), so the second graph pictured is the only possibility (cycles being forbidden by (3)). Finally, let Γ have only single edges; if Γ has no node, it must be a simple chain (again because no cycles are allowed). It cannot contain more than one node (7), so the fourth graph is the only remaining possibility.

(9) *The only connected* Γ *of the second type in* (8) *is the Coxeter graph* F_4  *or the Coxeter graph* $\mathsf{B}_n(=\mathsf{C}_n)$  .

Set $\varepsilon = \sum_{i=1}^{p} i\varepsilon_i$, $\eta = \sum_{i=1}^{q} i\eta_i$. By hypothesis, $2(\varepsilon_i, \varepsilon_{i+1}) = -1 = 2(\eta_i, \eta_{i+1})$, and other pairs are orthogonal, so $(\varepsilon, \varepsilon) = \sum_{i=1}^{p} i^2 - \sum_{i=1}^{p-1} i(i+1) = p(p+1)/2$, $(\eta, \eta) = q(q+1)/2$. Since $4(\varepsilon_p, \eta_q)^2 = 2$, we also have $(\varepsilon, \eta)^2 = p^2q^2(\varepsilon_p, \eta_q)^2 = p^2q^2/2$. The Schwartz inequality implies (since ε, η are obviously independent) that $(\varepsilon, \eta)^2 < (\varepsilon, \varepsilon)(\eta, \eta)$, or $p^2q^2/2 < p(p+1)q(q+1)/4$, whence $(p-1)(q-1) < 2$. The possibilities are: $p = q = 2$ (whence F_4) or $p = 1$ (q arbitrary), $q = 1$ (p arbitrary).

(10) *The only connected* Γ *of the fourth type in* (8) *is the Coxeter graph* D_n

 *or the Coxeter graph* E_n $(n = 6, 7$ *or* $8)$

. Set $\varepsilon = \Sigma\, i\varepsilon_i$, $\eta = \Sigma\, i\eta_i$, $\zeta = \Sigma\, i\zeta_i$. It is clear that ε, η, ζ are mutually orthogonal, linearly independent vectors, and that ψ is not in their span. As in the proof of (4) we therefore obtain $\cos^2\theta_1 + \cos^2\theta_2 + \cos^2\theta_3 < 1$, where $\theta_1, \theta_2, \theta_3$ are the respective angles between ψ and ε, η, ζ. The same calculation as in (9), with $p-1$ in place of p, shows that $(\varepsilon, \varepsilon) = p(p-1)/2$, and similarly for η, ζ. Therefore $\cos^2\theta_1 = (\varepsilon, \psi)^2/(\varepsilon, \varepsilon)(\psi, \psi) = (p-1)^2(\varepsilon_{p-1}, \psi)^2/(\varepsilon, \varepsilon) = \frac{1}{4}(2(p-1)^2/p(p-1)) = (p-1)/2p = \frac{1}{2}(1-1/p)$. Similarly for θ_2, θ_3. Adding, we get the inequality $\frac{1}{2}(1-1/p+1-1/q+1-1/r) < 1$, or (*) $1/p+1/q+1/r > 1$. (This inequality, by the way, has a long mathematical history.) By changing labels we may assume that $1/p \le 1/q \le 1/r$ ($\le 1/2$; if p, q, or r equals 1, we are back in type A_n). In particular, the inequality (*) implies $3/2 \ge 3/r > 1$, so $r = 2$. Then $1/p + 1/q > 1/2$, $2/q > 1/2$, and $2 \le q < 4$. If $q = 3$, then $1/p > 1/6$ and necessarily $p < 6$. So the possible triples (p, q, r) turn out to be: $(p, 2, 2) = \mathsf{D}_n$; $(3, 3, 2) = \mathsf{E}_6$; $(4, 3, 2) = \mathsf{E}_7$; $(5, 3, 2) = \mathsf{E}_8$.

The preceding argument shows that the connected graphs of admissible sets of vectors in euclidean space are all to be found among the Coxeter graphs of types A–G. In particular, the Coxeter graph of a root system must be of one of these types. But in all cases except B_ℓ, C_ℓ, the Coxeter graph uniquely determines the Dynkin diagram, as remarked at the outset. So the theorem follows. □

Exercises

1. Verify the Cartan matrices (Table 1).
2. Calculate the determinants of the Cartan matrices (using induction on ℓ for types A_ℓ–D_ℓ), which are as follows:

$$A_\ell\colon \ell+1;\ B_\ell\colon 2;\ C_\ell\colon 2;\ D_\ell\colon 4;\ E_6\colon 3;\ E_7\colon 2;\ E_8,\ F_4 \text{ and } G_2\colon 1.$$

3. Use the algorithm of (11.1) to write down all roots for G_2. Do the same for C_3: $\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}$.
4. Prove that the Weyl group of a root system Φ is isomorphic to the direct product of the respective Weyl groups of its irreducible components.
5. Prove that each irreducible root system is isomorphic to its dual, except that B_ℓ, C_ℓ are dual to each other.
6. Prove that an inclusion of one Dynkin diagram in another (e.g., E_6 in E_7 or E_7 in E_8) induces an inclusion of the corresponding root systems.

Notes

Our proof of the classification theorem follows Jacobson [1]. For a somewhat different approach, see Carter [1]. Bourbaki [2] emphasizes the classification of Coxeter groups, of which the Weyl groups of root systems are important examples.

12. Construction of root systems and automorphisms

In §11 the possible (connected) Dynkin diagrams of (irreducible) root systems were all determined. It remains to be shown that each diagram of type A–G does in fact belong to a root system Φ. Afterwards we shall briefly discuss Aut Φ. The existence of root systems of type $A_\ell - D_\ell$ could actually be shown by verifying for each classical linear Lie algebra (1.2) that its root system is of the indicated type, which of course requires that we first prove the semisimplicity of these algebras (cf. §19). But it is easy enough to give a direct construction of the root system, which moreover makes plain the structure of its Weyl group.

12.1. Construction of types A–G

We shall work in various spaces $\mathbf{R}^n$, where the inner product is the usual one and where $\varepsilon_1, \ldots, \varepsilon_n$ denote the usual orthonormal unit vectors which