Exercise 5: Some things about weights and representations.

(1) Let $\mathfrak{g}$ be a finite-dimensional complex semisimple Lie algebra.
  (a) Show that if $L(\lambda)$ and $L(\mu)$ are highest weight modules (of weights λ and μ), show that $L(\lambda) \otimes L(\mu)$ has $L(\lambda + \mu)$ as a submodule with multiplicity 1. (Think about primitive elements)
  (b) Show that 0 is a weight of highest weight module $L(\lambda)$ if and only if λ is a sum of roots.

(2) **Type A_r stuff.** Analyze the standard representation of $\mathfrak{sl}_3$.
  (a) What are the primitive elements?
  (b) What is/are the weight/weights of the action of $\mathfrak{h}$ on the primitive elements (in terms of ω_1 and ω_2)?
  (c) What is the standard representation isomorphic to (in terms of highest weight modules)?
  (d) Draw a picture of the weights and verify that the dimension is correct.
  (e) What is the standard representation (in terms of highest weight modules) of $\mathfrak{sl}_{r+1}$ in general?

(3) **Type C_r stuff.**
  (a) Give a base for the set of roots of type C_r, and calculate the corresponding simple co-roots and fundamental weights.
  (b) Give two examples of highest weight modules for C_2 for which every weight space has multiplicity 1 (and justify how you know every weight space has multiplicity 1).