# Math 126 Numerical PDEs: Homework 6—debriefing

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1. [8 pts = 2+3+3]

   (a) Your switch can just test if $k$ is zero or not (this is not generally a good idea with floating-point numbers, but here it's ok).

   (b) At $N = 50$ errors inside are of order $10^{-4}$, worse within 0.1 distance of boundary. This is much worse than for Laplace, but, still acceptable if 4 digits is enough.

   (c) Convergence is algebraic, 3rd-order as Jeff showed. Why? Need to look up analytic properties of Hankel functions, *e.g.* `dlmf.nisg.gov`, or Abramowitz & Stegun handbook on functions, 10th edition, Ch. 9.1: $H_n(z) = J_n(z) + iY_n(z)$ with $J_1(z)$ a power series of the form $a_1 z + a_3 z^3 + a_5 z^5 + \ldots$, and $Y_1(z)$ of the form $b_{-1}z^{-1} + b_1 z + b_3 z^3 + \cdots + c\ln(z/2)J_1(z)$. The kernel of the DLP is proportional to $H_1(k|y(s) - y(t)|)\cos(\theta_{st})|\dot{y}(t)|$. As with Laplace, the $\cos(\theta_{st})$ term increases all the powers of $(s - t)$ by one. But due to the log terms, the kernel is *not* analytic at $s = t$, rather it has a very weak singularity of leading form $(s - t)^2 \ln|s - t|$. Now, a log singularity alone leads to 1st-order convergence since its Fourier coefficients die like $1/|n|$ (it is the imaginary partner to the step-function). The power of two makes this 3rd-order. Note the kernel is still continuous, in fact in $C^1([0, 2\pi)^2)$, so Nyström still provably converges.

2. [6 pts = 4+2]

   (a) See Brad's picture. Note you can make the y-coordinate increase upwards using `imagesc(x,y,data); set(gca,'ydir','normal');`

   (b) To 4 digit accuracy, the value at $(-2, 0)$ is $1.2762 + 0.0752i$. This was quite hard to get since roughly $N = 400$ is needed to ensure the 3rd-order algebraic convergence has achieved the desired accuracy. Be sure to add the incident and scattered potentials.

3. [7 pts = 3+1+2+1] Some nice plots here, eg see Lin for min sing val, or Jeff for blow-up demo.

   (a) By zooming in, or minimizing the function, you can get $k_1 = 2.704759\ldots$ for lowest Dirichlet eigenvalue. Not you'll need to increase $N$ to get this accurate or moreso.

   (b) By choosing generic (e.g. random) boundary data $f$ you will see the 'physical' (ie BVP) interior solution blows up as $k \to k_1$. (However if you use Dirichlet data from the solution potential in question #1, you won't get blow-up since this solution doesn't blow up. In fact one can prove via the GRF that this data is orthogonal to the blowing-up subspace).

   (c) $k_1^{(N)} = 1.51067\ldots$ is lowest nonzero Neumann eigenvalue.

   (d) $u_s$ does not blow up as $k \to k_1^{(N)}$, so this is purely a numerical (i.e. BIE representation) problem. This last issue motivates the combined-field (CFIE) representation $\mathcal{D} - i\eta\mathcal{S}$.