Math 116 Numerical PDEs: Homework 4—debriefing

February 8, 2012

Tips:

Please answer all the questions asked - each is worth something even if only one point!

Don't make your outer ('driver') code a function. Just leave it a script. That way you have access to all the variables.

Choose meaningful names for functions, not just 'q5helper.m' since you'll never remember what it does later. You might (probably will) need it later!

1. [6 pts = 3+3]

   (a) Either use q_{n+1} orthog to $\mathbb{P}_n$ as the defining property of a Gauss quad and create a new poly with simple roots where q_{n+1} changes sign (Lin + Brad + Vipul + Jeff), or use the property that it integrates all $\mathbb{P}_{2n+1}$ exactly (Taylor).

   (b) Note two integrals required. See Taylor proof. The 2-norm of the kernel function on the square is called the *Hilbert-Schmidt norm*, and when finite, the operator itself is Hilbert-Schmidt class.

2. [3+1+1 = 5 pts] Mostly fine. See e.g. Brad's.

3. [13 pts: 7 + 2 + 4] See Jeff, who gave explicit formulae for linear xform of nodes and weights to arbitrary interval.

   (a) Generally good!

   (b) Condition number tends to about 2.4 as N grows. This is because of the identity part, i.e. the 2nd-kind nature of the IE. Note this is *much better* than some of the matrices we dealt with before where cond blows up exponentially! (this would apply to 1st-kind too).

   (c) Looking at the graph of the error function, you can see it never gets much bigger than the worst error at the nodes.

   If you want, I have solution code `hw4ans3c_nystrom_interpolation.m`

4. [8 pts = 2+2+1+3] See e.g. Brad's solution, but Jeff for discussion in d, and Taylor for proof in c. The tricky thing in this question is keep track of signs and 2π's. You should have a consistent Fourier series convention, i.e. if $u(s) = \sum_{m\in\mathbb{Z}} u_m e^{-ims}$ then its inverse is $u_m = (1/2\pi)\int_0^{2\pi} u(s)e^{ims}ds$.

   (a) The trick is changing variable. Get eigenvalue of $\tilde{k}_{-m}$.

   (b) If you are careful with your signs you get $\tilde{k}_m u_m = f_m$, $\forall m \in \mathbb{Z}$. This is simply a multiplication operator!

   (c) Just as in finite-dim space, for a diagonal matrix, the largest magnitude eigenvalue (diagonal entry) gives the norm. See Taylor (apart from 2π factor).

   (d) Eigenvalues of K^{-1} are the inverses of those of K. By the way, Parseval is $\int_0^{2\pi} |\tilde{k}(s)|^2 ds = 2\pi \sum_{n\in\mathbb{Z}} |\tilde{k}_n|^2$.

   However, my question was slightly loose here: really I was after an upper bound on the relative condition number (if it exists), as we did in finite-dim linear systems. (The relative condition number depends on the input data.) I.e. we showed $\kappa = \|A\|\,\|A^{-1}\|$ is an upper bound on the relative condition number of the problem $A\mathbf{x} = \mathbf{b}$, and the same is true for the operator, with A replaced by K.

For 1st-kind, the condition number is always infinite because

$$\|K^{-1}\| = \sup_{m\in\mathbb{Z}} |\tilde{k}_m|^{-1} = \infty$$

but for 2nd-kind, it is bounded by

$$\|I - K\| \, \|(I - K)^{-1}\| = \sup_{m\in\mathbb{Z}} |1 - \tilde{k}_m| \cdot \sup_{m\in\mathbb{Z}} |1 - \tilde{k}_m|^{-1}$$

which is finite unless (as Jeff discusses) there is a kernel coefficient (i.e. eigenvalue) $\tilde{k}_m = 1$ for some m.

5. [5pts = 3+2] E.g. see Brad's.

   (a) Your expression should be $\frac{1}{2\pi} \frac{n\cdot(x-y)}{|x-y|^2}$.

   (b) Contours are circles tangent to a common line at the point y. You can prove it. Use `axis equal;` to show they are not ellipses.