Math 116 Numerical PDEs: Homework 4

due Mon Feb 6, 9am

Please heed the advice in the HW3 debriefing. Spread out the work and seek help or collaborate before spending lots of time stuck. A mix of analysis and coding, apt for a computational mathematician.

1. [quick analysis ones]
   (a) Prove that all of the roots of polynomial which defines the nodes for an $(n+1)$-node Gaussian quadrature are simple—we never showed this in lecture. [Hint: Look at the proof in notes showing you can't integrate exactly all polynomials of degree $2n+2$.]
   (b) Prove that the 2-norm of an integral operator K is bounded by the 2-norm of its kernel function on the square $[a,b]^2$. [Hint: Cauchy-Schwarz]

2. Solve analytically the second-kind integral equation,

$$u(t) + \int_0^1 ts^3 u(s)ds \;=\; 1, \qquad \text{for } t \in [0,1] \tag{1}$$

   [Hint if stuck: u is the RHS plus something in the range of K, the integral operator]. Compute $\|K\|_\infty$. Is K compact, and why?

3. [the main one] Code up the 1D Nyström method in a way that allows you to switch easily between different quadrature schemes (*e.g.* by setting a switch variable at the start of your code). Apply it to the second-kind Fredholm equation

$$u(t) + \int_0^1 e^{ts} u(s)ds \;=\; e^t + \frac{1}{t+1}(e^{t+1}-1) \qquad \text{for } t \in [0,1] \tag{2}$$

   which you can check has unique solution $u(t) = e^t$.
   (a) Produce plots that show the convergence vs N, the number of nodes, of the maximum error magnitude in u over the nodes, for the two schemes: i) composite trapezoid, and ii) Gaussian quadrature. (If you like, $N = n+1$ since we labeled our nodes 0 to n for these schemes in lecture.) Categorize the convergence in each case and relate it to that of the quadrature scheme. What N is required in each case to reach an error smaller than 10^{-5}?
   (b) How does the condition number of the linear system you are solving change with N? (You don't need to plot this, just describe).
   (c) At $N = 5$ for Gaussian quadrature, produce a plot of the difference between the Nyström interpolated solution *function* $u_n(t)$ and the exact solution, on a fine grid on the interval $[0,1]$. (Don't show the two functions, just subtract them). Overlay the errors at just the 5 nodes onto your graph as blobs. Is the true error sup norm of the solution reflected by the maximum error magnitude in u over the nodes, as you assumed in the part (a)?

4. Here you explore analytically Fredholm equations involving a "periodic convolution operator", that is, an operator acting on functions on $[0,2\pi)$ with kernel of the form $k(s,t) = (1/2\pi)\tilde{k}(t-s)$, where $\tilde{k} : \mathbb{R} \to \mathbb{C}$ is a 2π-periodic function. They also have applications in signal and image processing. You will show that they become very simple to solve in the Fourier basis.

(a) Let K be such an operator. Show that e^{imt}, for any $m \in \mathbb{Z}$, is an eigenfunction of K, and find its eigenvalue λ_m.

(b) By substituting a Fourier series $f(s) = \sum_{m\in\mathbb{Z}} f_m e^{-ims}$ and similar for u and $\tilde{k}$, convert the first-kind Fredholm equation $Ku = f$ into a set of simple algebraic relations involving the Fourier coefficients $\{f_m\}$, $\{u_m\}$ and $\{\tilde{k}_m\}$. [Hint: you'll need orthogonality of $\{e^{imt}\}$ on $[0, 2\pi)$]

(c) What is $\|K\|_2$? [Hint: go into a Fourier basis and use (b)]

(d) If $\tilde{k}$ is in $L^2(0, 2\pi)$ then its Fourier coefficients decay as $|m| \to \infty$, by Parseval's equality. What then is the *condition number* of the 1st-kind problem $Ku = f$? [Hint: what does K^{-1} do?] What is the condition number of the 2nd-kind problem $u - Ku = f$? BONUS: What also can you say about compactness of K?

5. The fundamental solution for Laplace's equation in 2D is $\Phi(x, y) = (1/2\pi)\ln 1/|x - y|$, where y is a source point in $\mathbb{R}^2$, and x a target point also in $\mathbb{R}^2$. Here you examine its directional derivative, a "dipole source".

(a) Make a function which returns $\partial\Phi(x, y)/\partial n_y$, the derivative with respect to source location in the direction n_y, given vectors $x, y \in \mathbb{R}^2$ and a unit vector $n_y \in \mathbb{R}^2$. Generalize your routine so that it handles multiple x vectors (*e.g.* a 2-by-n matrix of coordinates of n such vectors), and returns the corresponding list of outputs. (Be sure to test it on known inputs!)

(b) Use the above to produce a contour plot of $\partial\Phi(x, y)/\partial n_y$ for $y = (0.5, -0.2)$, $n_y = (1/2, \sqrt{3}/2)$, for x varying over the square $[-1, 1]^2$. This should now be a 3-line program.