Math 116 Numerical PDEs: Homework 3

due Mon 9am, Jan 30

Complex analysis keeps coming up this week, crucial to understand quadrature convergence!

1. Visualizing & understanding functions on $\mathbb{C} \to \mathbb{C}$

   (a) Make a 3D height plot of absolute value of $f(z) = (1 + 25z^2)^{-1}$ over the complex box Re z, Im $z \in [-1, 1]$. Choose a good vertical scale. Where, and of what type, are any singularities?
   [Hint: look up `[xx, yy] = meshgrid ...` then use `zz = xx+1i*yy;`]

   (b) Do the same for $f(z) = e^{-1/z}$. (Ask if stuck about the singularity type.)

   BONUS Get `z2rgb_kawski.m` from the website and try the following given the `zz` data from (a) or (b):
   `surf(xx, yy, 0*xx, z2rgb_kawski(zz)); view(2); shading interp`
   Black is vanishing size and white is size ∞, but what new information is shown?

2. Prove that, given a set of distinct points $\{x_j\}_{j=0,\ldots n}$ in $[a, b]$ there exists a *unique* set of weights $\{w_j\}_{j=0,\ldots,n}$ such that Newton-Cotes quadrature integrates exactly over $[a, b]$ all polynomials up to degree n. Use a different method than the two in lecture. [Hint: proof is one line. Write the equations the weights must satisfy for the monomial basis for $\mathbb{P}_n$.]

3. Consider numerical integration of $(1 + 4x^2)^{-1}$ on $[-1, 1]$ (note: exact answer easy). Compare with suitable plots the convergence vs $n = 1, \ldots, 40$ of the quadrature error using $n + 1$ equally-spaced nodes using the following two schemes for weights:

   (a) Composite trapezoid rule. Choose axes so that the convergence is a straight line. Does it match the theorem from lecture?

   (b) Newton-Cotes, *i.e.* interpolatory quadrature. Here you will need to get the weights from solving a simple linear system. [Hint: see previous question.] What is the minimum-achievable error?

4. Get from the website `gauss.m` which provides nodes and weights of $(n+1)$-node Gaussian quadrature on $[-1, 1]$. Use this to produce convergence plots of quadrature error for the integrals of the following functions on $[-1, 1]$. Note, in each case you can compute the analytic answer to compare to.

   (a) $(1 + 4x^2)^{-1}$ (is performance better than in question #3?)

   (b) x^{20}

   (c) $|x|^3$

   Measure the order (if algebraic) or rate (if exponential, i.e. give α in $Ce^{-\alpha n}$) of convergence in each case, or state another kind of behavior, as appropriate. Discuss reasons for what you observe—in one case you should have a theorem, but in the others just intuition is fine (theorems there require extra reading and are thus BONUS).

5. We discussed that given an inner product $(\cdot, \cdot)$ on $L^2[-1, 1]$ a sequence (q_n) of orthogonal polynomials can be constructed by applying the Gram-Schmidt procedure from linear algebra to the monomials $1, x, x^2, \ldots$. Prove that the following 3-term recurrence relation also constructs them:

$$
\begin{aligned}
q_{-1}(x) &= 0 \\
q_0(x) &= 1 \\
q_{j+1}(x) &= xq_j(x) - \alpha_{j+1}q_j(x) - \beta_{j+1}q_{j-1}(x), \qquad j = 0, 1, 2, \ldots
\end{aligned}
$$

where $\alpha_{j+1} := (q_j, xq_j)/(q_j, q_j)$ and $\beta_{j+1} := (q_j, q_j)/(q_{j-1}, q_{j-1})$ with the exception of $\beta_1 = 0$. [Hint: one approach is to start with Gram-Schmidt and notice all but two projection terms vanish. Another is to prove that the above recurrence generates a *mutually* orthogonal set, i.e. orthogonal to all lower poly's. To make life easier, use a shorthand notation such as $(xj, j+1)$ for (xq_j, q_{j+1}).]

6. Compare error convergence (`semilogy` plots) for the integrals over $[0, 2\pi)$ of the following three 2π-periodic functions using the periodic trapezoid rule, in each case explaining how analyticity (where are singularities? qu. #1 may help) controls convergence rate:

   (a) $(1/2\pi)e^{\cos x}$. The exact answer is the modified Bessel function $I_0(1)$ which in Matlab is `besseli(0,1)`.

   (b) $\left(1 + \cos^2(x/2)\right)^{-1}$. Since exact answer not easy, just use the value found once converged. Please quote this to 15 digits. Also, please add to your plot the predicted best rate from Thursday's lecture, which should agree.

   (c) $\exp(-1/|\sin(x/2)|)$. Go out to $n = 300$, ignore the 'scalloping' effect and instead look at overall behavior.

   [Hints: The coding is very easy here. One of the functions is C^∞ but not real analytic on $[0, 2\pi)$. Recall that the composition of entire functions is entire. Also you may need $\sin(ix) = i \sinh(x)$.]