Math 126 Numerical PDEs, Winter 2012: Homework 1

due Monday 9am Jan 16

Ideally you should start to use LaTeX to write this up, then post it to your webpage with your codes. A good thing to do this weekend is to get both those set up (make a baby webpage and a baby LaTeX document.) Codes should be short with explanatory comments if needed. See honor code in syllabus.

1. If $\mathbf{u}$ and $\mathbf{v}$ are m-vectors, the matrix $A = I + \mathbf{u}\mathbf{v}^*$ is known as a *rank-one perturbation of the identity.* Show that if A is non-singular, then its inverse has the form $A^{-1} = I + \alpha\mathbf{u}\mathbf{v}^*$ for some scalar α, and give an expression for α. For what $\mathbf{u}$ and $\mathbf{v}$ is A singular? If it is singular, what is Nul A? [NLA Ex 2.6]

2. The spectral radius $\rho(A)$ of a square matrix A is the magnitude of its largest eigenvalue. Prove that $\rho(A) \le \|A\|_2$. [NLA Ex 3.2]

3. Use the in-class worksheet on the following $m \times m$ bidiagonal matrix to answer the below.

$$A = \begin{bmatrix} 1 & 2 & & \\ & 1 & 2 & \\ & & 1 & \ddots \\ & & & \ddots \end{bmatrix}$$

   (a) find a nontrivial lower bound on the condition number $\kappa(A)$
   (b) predict the smallest m such that roughly all significant digits will be lost in the solution $\mathbf{x}$ to a linear system $A\mathbf{x} = \mathbf{b}$ in double precision.
   (c) demonstrate your last claim in a couple of lines of code, by starting with a known $\mathbf{x}$, computing $\mathbf{b}$ then solving via `mldivide`. [Hint: look up the `toeplitz` command to construct the matrix rather than use a loop. You need to choose a $\mathbf{b}$ that causes floating-point rather than exact integer arithmetic to be used!]

4. How many nested loops are implied by each of the following MATLAB commands? (*i.e.* how many loops would you need to write to code the equivalent in C or fortran?) `A = rand(100,100); x = 1:100; b = A*x'; B = A*A;`

5. Give an exact formula, in terms of β and t, for the smallest positive integer n that does not belong to the floating-point system $\mathbf{F}$, and compute n for IEEE single- and double-precision. Give one line of code, and its output, which demonstrates this is indeed the case for double-precision. [NLA Ex 13.2]

6. Measure how the time to compute the singular values of a random real dense $m \times m$ matrix scales with m, focusing on the range $10^2 \le m \le 10^3$. Produce a log-log graph of time vs m, and the simple power law to which it is asymptotic. BONUS: for what large m would you expect this to break down and why?

7. Consider the polynomial $p(x) = (x-2)^9 = x^9 - 18x^8 + 144x^7 - 672x^6 + 2016x^5 - 4032x^4 + 5376x^3 - 4608x^2 + 2304x - 512$. [NLA Ex 13.3]

   (a) plot $p(x)$ for $x = 1.920, 1.921, 1.922, \ldots, 2.080$ evaluating p via its coefficients $1, -18, 144, \ldots$.
   (b) overlay on your plot the same computed using $(x-2)^9$
   (c) explain, including the size of the effect!