Math 124 Fall 2005

Current problems in Topology

Final Exam

Thursday December 1, 2005

The exam is due at 11:50 PM Wednesday December 7, 2005
in the Instructor's office 414 Bradley Hall.

Your name (please print): _____

Instructions: This is an open book open notes exam. You can consult any printed matter you like, but **you can not consult other humans. Use of calculators is not permitted**. **You must justify all of your answers to receive credit,** unless instructed otherwise. If I am not in my office when you submit your exam, then please write the time you finished working on it and slide it under my office door.

The exam total score is the sum of the 10 best (out of 11) problem scores.

The Honor Principle requires that you neither give nor receive any aid on this exam.

Grader's use only

1. _____ /15

2. _____ /15

3. _____ /15

4. _____ /15

5. _____ /15

6. _____ /15

7. _____ /15

8. _____ /15

9. _____ /15

10. _____ /15

11. _____ /15

Total: _____ /150

(1) Let $\mathbf{X}(x, y, z) = x\frac{\partial}{\partial x}$ be a vector field on $\mathbb{R}^3$. Let $\omega(x, y, z) = xdx + 2dy + 3dz$ be 1-form on $\mathbb{R}^3$. Compute the Lie derivative $L_{\mathbf{X}}\omega$ at the point $(3, 0, 0)$.

(2) Show that the bundle $\Omega^2(TM) \to M$ of the alternating 2-tensors over $M = S^1 \times S^1 \times S^1$ is trivial.

(3) Let ω be a closed differential 1-form on $\mathbb{R}P^3$ and let $c : S^1 \to \mathbb{R}P^3$ be an embedding. Prove that $\int_{S^1} c^*(\omega) = 0$.

(4) Let $c : S^1 \to S^1 \times S^3$, $f : S^1 \times S^3 \to S^2 \times S^2$ and $g : S^2 \times S^2 \to \mathbb{R}P^3 \times S^1$ be C^∞-mappings. Let w be a closed 1-form on $\mathbb{R}P^3 \times S^1$. Compute $\int_{S^1}(g \circ f \circ c)^*(\omega)$. Here all the spheres S^k are oriented as the boundaries of the balls B^{k+1}, that in turn are oriented as the subspaces of $\mathbb{R}^{k+1}$ equipped with the standard orientation.

(5) Let $M = \{(x, y, z) \in \mathbb{R}^3, x, y, z > 0\}$. Let $\mathbf{X_1}(x, y, z) = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ and $\mathbf{X_2}(x, y, z) = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}$ be vector fields on M.

a: Is it possible to find a coordinate system (U, t) around $(1, 2, 3) \in M$ in which the vector fields $\mathbf{X_1}, \mathbf{X_2}$ look as $\frac{\partial}{\partial t_1}$ and $\frac{\partial}{\partial t_2}$ respectively? (Prove your answer.)

b: Is it possible to find an integral submanifold to the distribution $\Delta_{(x,y,z)} = \mathrm{Span}\big(\mathbf{X_1}_{(x,y,z)}, \mathbf{X_2}_{(x,y,z)}\big)$ on M that passes through the point $(1, 2, 3)$? (Prove your answer.)

(6) Let $p : S^2 \to \mathbb{R}P^2$ be the double cover. Let ω be a 2-form on $\mathbb{R}P^2$. Prove that $p^*(\omega)$ is zero at at least two points of S^2.

(7) Which one of the following can be the boundary of an oriented manifold N. If an item is a boundary of N, explain what N is. If an item can not be a boundary of an oriented N, explain why.

1: $S^1 \times S^2$
2: $S^1 \times \mathbb{R}P^4$
3: $\mathbb{R}P^1$
4: $\mathbb{R}P^2$
5: wedge of S^1 and S^1
6: connected sum of S^1 and S^1.

(8) Let $M = M_1 \times M_2$ be a product of two manifolds with boundary. Show that $\partial(M_1 \times M_2) = \big((\partial M_1) \times M_2\big) \cup \big(M_1 \times \partial(M_2)\big)$. For simplicity assume that all the manifolds are continuous rather than C^∞.

(9) Let $\mathbf{X}(x, y, z) = e^{xyz}\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}$ be a vector field on $\mathbb{R}^3$ and $f(x, y, z) = x + 2y + 3z$ be a function. Compute the Lie derivative $L_{\mathbf{X}}f$ at the point $(1, 0, 1)$.

(10) Assume that $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ are C^∞ vector field on S^{17}. Assume that $L_{\mathbf{X}}(Y) = 0$ (is identically zero as a vector field), and that $[\mathbf{Y}, \mathbf{Z}] = \mathbf{X} + 2\mathbf{Y}$. Prove that $L_{\mathbf{Y}}([\mathbf{Z}, \mathbf{X}]) = 0$. (Hint: you might want to use some basic Lie algebra facts.)

(11) Let $M = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 - x_3 < 0\}$ be the 3-manifold which is the filled open paraboloid. Compute the dimension of the quotient space of the vector space of all closed 3-forms on M by its vector subspace formed by all the exact 3-forms on M.