V63.0123-1 : Calculus III. Midterm1

Wed Feb 19. You have 60 minutes. Potentially useful equations are on back.

1. [10 points]

   Find the equation for the plane which includes the point $P = (1, 4, -2)$ and the line $x = 2 + t$, $y = -5t$, $z = 1 - 3t$.

2. [14 points]

   A point particle moves according to the two-dimensional vector function $\mathbf{r}(t) = (e^t \cos t,\ e^t \sin t)$.

   (a) Compute the velocity and speed of this particle at $t = 0$.

   (b) Show that this particle crosses the coordinate axes when t is an integer multiple of $\pi/2$, and when this happens, the trajectory makes an angle of $45°$ with the coordinate axis it crosses.

   (c) Make a rough sketch of the trajectory for $t > 0$.

   (d) Compute the distance the particle covers between $t = 0$ and $t = 2$.

3. [9 points]

   Someone holds a laser pointer at position $(1, 2, -1)$ and shines the beam (a straight line) parallel to the vector $(1, -1, 2)$. Your eye is located at the origin. Does the beam hit you in the eye? If so, explain. If not, find the closest distance the beam passes to your eye.

4. [8 points]

   Sketch the object defined in cylindrical coordinates by $z = 2-r$, with $z > 0$ and $\theta \in [0, 2\pi]$. Label any intersections with the axes. [Hint: first sketch the object $z = -r$ then proceed to $z = 2 - r$].

5. [9 points]

   Find the position vector of a particle at a general time t if the particle starts at $P = (3, 3, 3)$ at $t = 0$ and has velocity given by the vector function $\left(\frac{1}{1+t},\ 1 + t,\ -\sin 2t\right)$. Express your answer in terms of the unit vectors $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$.

$$
\begin{aligned}
\sin(\theta+\phi) &= \sin\theta\cos\phi + \cos\theta\sin\phi \\
\cos(\theta+\phi) &= \cos\theta\cos\phi - \sin\theta\sin\phi \\
\sin^2\theta &= \tfrac{1}{2}(1-\cos 2\theta) \\
\cos^2\theta &= \tfrac{1}{2}(1+\cos 2\theta) \\
\frac{d}{dx}\sin x &= \cos x \\
\frac{d}{dx}\cos x &= -\sin x \\
\frac{d}{dx}\tan x &= \sec^2 x \\
\frac{d}{dx}\sin^{-1} x &= \frac{1}{\sqrt{1-x^2}} \\
\frac{d}{dx}\cos^{-1} x &= -\frac{1}{\sqrt{1-x^2}} \\
\frac{d}{dx}\tan^{-1} x &= \frac{1}{1+x^2} \\
\text{surface area of revolution} &= \int_a^b dt\, 2\pi y(t)\sqrt{x'(t)^2+y'(t)^2} \\
\kappa &= \frac{|\mathbf{T}'|}{|\mathbf{r}'|} = \frac{|\mathbf{r}''\times\mathbf{r}'|}{|\mathbf{r}'|^3} \\
\text{spherical coords:}\quad x &= \rho\sin\phi\cos\theta \\
y &= \rho\sin\phi\sin\theta \\
z &= \rho\cos\phi
\end{aligned}
$$