Calculus III - Midterm - March 20, 2002

1. Let $z = f(x^2 - y^2)$ for some differentiable function f. Show that

$$y\,\frac{\partial z}{\partial x} + x\,\frac{\partial z}{\partial y} = 0.$$

(Hint: Let $u = x^2 - y^2$) **(15 pts)**

2. Find an equation of the plane tangent to the surface $x^2y + y^2z + z^2x = 3$ at the point $(1, 1, 1)$. (Hint: gradient) **(15 pts)**

3. Let $f(x, y) = \sqrt{xy}$. Find a linear approximation to f of the form $L(x, y) = ax + by + c$ at the point $(1, 4)$. Then evaluate $L(1.1, 4.1)$ to find an approximate value for $\sqrt{(1.1)(4.1)}$. **(15 pts)**

4. The equation of a surface is given in the cylindrical coordinates to be $z = 4 - r$, $0 \leq \theta \leq 2\pi$. Sketch the portion of the graph of this surface that lies above the xy plane. **(15 pts)** (Hint: First plot the surface $z = r$, then $z = -r$, and finally $z = 4 - r$.)

5. Let $f(x, y) = \dfrac{x}{y} + \dfrac{y}{x}$. What is the domain of f?
(a) Show that in its domain, f is constant on any line that passes through the origin.
(b) Compute the gradient $\vec{\nabla} f$ at a given point (r, s) and show that it is perpendicular to the vector $\vec{\mathbf{v}} = (r, s)$. Can you deduce the result in (a) directly from this?
(Hint: What is the value of the directional derivative of f along $\vec{\mathbf{v}}$?)
(c) Does $\lim_{(x,y)\to(0,0)} f(x, y)$ exist? If yes, compute the limit. If not, give your reasoning.
(20 pts)

6. A point particle moves according to the vector function $\vec{\mathbf{r}}(t) = (e^t \cos t, e^t \sin t)$, starting at time $t = 0$.
a) Compute the initial velocity and speed of this particle.
b) Show that this particle crosses the coordinate axes when t is an integer multiple of $\frac{\pi}{2}$ and when this happens, the trajectory makes an angle of 45 degrees with the coordinate axis it crosses.
c) Make a rough sketch of the trajectory.
d) Compute the distance that this particle covers between $t = 0$ and $t = 2$.
(20 pts)

Extra credit:

7. Find an equation of the plane defined by the triangle with vertices $A(1, 1, 1)$, $B(2, 0, 0)$, and $C(0, 2, 0)$. Sketch this triangle. **(10 pts)**