Induced Charges on a Dielectric Sphere in a Constant Electric Field

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Abstract

A dielectric sphere (dielectric constant ϵ_1) of radius R is placed into space with dielectic constant ϵ_2. There is an external homogeneous electric field E. I expand the electric potential in spherical harmonics inside the sphere and outside of it, and, using boundary conditions I find the potential everywhere in space. Given the potential I derive the density of polarized charges on the boundary of the sphere. I will perform numerical calculation using Nystrom method to find numerical solution for charge density. Later, I will use analiticaly derived solution for density of polarized charges as a model and will compare numerical results with that model. However, problem that we are facing with is not preciesely the same as for dielectric sphere (in our case it's conducting sphere), so charges can move around the surface of a sphere. But for simplicity I choose the above mentiond approach.

1. ANALYTICAL SOLUTION

I work in spherical coordinates r, θ, ϕ. The electric field is in the direction of the $z-$axes. I use the following Maxwell's equations:

$$\nabla \times \mathbf{E} = 0, \tag{1}$$
$$\nabla\cdot\mathbf{E} = 0, \text{ for } r < R, \tag{2}$$
$$\nabla\cdot\mathbf{E} = 0, \text{ for } r > R, \tag{3}$$
$$\nabla\cdot\mathbf{E} \neq 0, \text{ for } r = R. \tag{4}$$

Eq. (1) implies that everywhere in space there exists a potential function $\Phi(\mathbf{r})$, such that:

$$\mathbf{E} = -\nabla\Phi(\mathbf{r}). \tag{5}$$

From Eqs. (2,3,5) it follows that:

$$\nabla^2\Phi(\mathbf{r}) = 0, \text{ for } r < R, \tag{6}$$
$$\nabla^2\Phi(\mathbf{r}) = 0, \text{ for } r > R. \tag{7}$$

Meaning of Eq. (4) is that there is some induced charge on the surface of the dielectric sphere ($r = R$), which I will calculate once we derive the potential function $\Phi(\mathbf{r})$ (see below).

I need to solve Eqs. (6,7) with the boundary conditions on the sphere ($r = R$) such that the tangential component of the electric field remains constant and the normal component changes discontinuously according to equations:

$$\mathbf{E}_\phi(R - 0^+, \theta, \phi) = \mathbf{E}_\phi(R + 0^+, \theta, \phi), \tag{8}$$
$$\mathbf{E}_\theta(R - 0^+, \theta, \phi) = \mathbf{E}_\theta(R + 0^+, \theta, \phi), \tag{9}$$
$$\epsilon_1\, \mathbf{E}_r(R - 0^+, \theta, \phi) = \epsilon_2\, \mathbf{E}_r(R + 0^+, \theta, \phi). \tag{10}$$

Note that our problem is completely symmetric under rotations around the $z-$axes. This means, that $E_\phi(\mathbf{r}) \equiv 0$. So, Eq. (8) is trivially satisfied by the symmetry of the problem. This also implies that the potential has no $\phi-$dependance: $\Phi(\mathbf{r}) \equiv \Phi(r, \theta)$. Thus, the remaining two boundary conditions (Eqs. (9,10)) in terms of the potential Φ look like:

$$\frac{\partial\Phi}{\partial\theta}(R - 0^+, \theta) = \frac{\partial\Phi}{\partial\theta}(R + 0^+, \theta), \tag{11}$$
$$\epsilon_1\, \frac{\partial\Phi}{\partial r}(R - 0^+, \theta) = \epsilon_2\, \frac{\partial\Phi}{\partial r}(R + 0^+, \theta). \tag{12}$$

The last obvious boundary condition is of course the continuity of the potential at the boundary $r = R$:

$$\Phi(R - 0^+, \theta) = \Phi(R + 0^+, \theta). \tag{13}$$

Now I need to solve Eqs. (6,7) with the boundary conditions Eqs. (11,12, 13).

It is well known that the solution of the Laplace's equation $\nabla^2\, \Phi(\mathbf{r}) = 0$ is:

$$\Phi(r, \theta, \phi) = \Sigma_{l=0}^{\infty}\Sigma_{m=-l}^{l}\, a_l^m\, \frac{1}{r^{l+1}}\, Y_l^m(\theta, \phi) + \Sigma_{l=0}^{\infty}\Sigma_{m=-l}^{l}\, b_l^m\, r^l\, Y_l^m(\theta, \phi), \tag{14}$$

where a_l^m and b_l^m are arbitrary constants, $Y_l^m(\theta,\phi)$ are spherical harmonic functions:

$$Y_l^m(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\,P_l^m(\cos\theta)\,e^{im\phi}, \tag{15}$$

where $P_l^m(\cos\theta)$ is the associated Legendre function.

Note that the first sum in Eq. (14) diverges as $r \to 0$ and goes to zero as $r \to \infty$. The second sum in that equation is finite as $r \to 0$ and starting from $l = 1$ term diverges as $r \to \infty$. Also note that since our potential $\Phi(r,\theta)$ doesn't depend on ϕ, we have to include Only $m = 0$ terms in Eq. (14). Taking these facts into account, I find that general physical solution to Eqs. (6,7) is:

$$\Phi(r,\theta) = \Sigma_{l=0}^{\infty} b_l\, r^l\, Y_l^0(\theta), \text{ for } r < R, \tag{16}$$

$$\Phi(r,\theta) = \Sigma_{l=0}^{\infty} a_l\, \frac{1}{r^{l+1}}\, Y_l^0(\theta) + A + B\, r Y_1^0(\theta), \text{ for } r > R. \tag{17}$$

A few notes about equations above. For $r < R$, I included only terms from the second sum of the Eq. (14), because all terms in the first sum of that equation diverge as $r \to 0$ which is unphysical, so these terms should not be in the potential. For $r > R$, I included all terms from the first sum of the Eq. (14), but also first two terms ($l = 0$ and $l = 1$) from the second sum of that equation. The reason is that $l = 0$ term of the second sum is just a constant, and $l = 1$ term is needed to reproduce the External Electric field E for $r >> R$.

Indeed, we know that for $r >> R$, $\Phi(\mathbf{r}) \approx -\mathbf{E}_\infty \cdot \mathbf{r} = -E_\infty\, r\cos\theta$. Thus, taking asymptotic expression of Eq. (17) as $r \to \infty$ we find $\Phi(r,\theta) \to B\, r\cos\theta$, so we immediately find that $B = -E_\infty$. (Here we used a well known identity $Y_1^0(\theta) = \cos\theta$).

Since the overall constant in the potential is unphysical, I can safely set $A = 0$ in Eq. (17). However choosing such A means I have fixed the overall additive constant in the potential, so I should keep all constants everywhere else.

Now let's rewrite Eqs. (16, 17) regrouping the corresponding terms Y_l^0 in such a way that the boundary conditions are easy to write in terms of a_l, b_l. Also, we plug in $A = 0$ and $B = -E_\infty$:

$$\Phi(r < R) = b_0 Y_0^0 + b_1 r\, Y_1^0 + \Sigma_{l=2}^{\infty}\, b_l\, r^l\, Y_l^0, \tag{18}$$

$$\Phi(r > R) = \frac{a_0}{r} Y_0^0 + \left(\frac{a_1}{r^2} - E_\infty\, r\right) Y_1^0 + \Sigma_{l=2}^{\infty} \frac{a_l}{r^{l+1}} Y_l^0. \tag{19}$$

now I plug in Eqs. (18,19) the boundary condition from Eq. (13):

$$b_0 = \frac{a_0}{R}, \tag{20}$$

$$b_1 R = \frac{a_1}{R^2} - E_\infty\, R, \tag{21}$$

$$b_l R^l = \frac{a_l}{R^{l+1}}, \text{ for } l \geq 2. \tag{22}$$

I used the orthonormality of the spherical basis here. It is clear that the boundary condition in Eq. (11) doesn't give us any new information.

Finally, plugging in the boundary condition in Eq. (12) into Eqs. (18,19) I get:

$$\epsilon_1 \cdot 0 = -\epsilon_2 \cdot \frac{a_0}{R^2}, \tag{23}$$

$$\epsilon_1 \cdot b_1 = -\epsilon_2 \cdot \left(\frac{2a_1}{R^3} + E_\infty \right), \tag{24}$$

$$\epsilon_1 \cdot b_l \, l \, R^{l-1} = -\epsilon_2 \cdot \frac{(l+1)a_l}{R^{l+2}}, \text{ for } l \geq 2. \tag{25}$$

Let's solve Eqs.(20-25):

From Eq.(23) I get $a_0 = 0$. Plugging this into (20) I find $b_0 = 0$.

Solving Eqs.(21,24) with respect to unknowns a_1, b_1 gives:

$$a_1 = -\frac{\epsilon_2 - \epsilon_1}{\epsilon_1 + 2\epsilon_2} E_\infty R^3, \tag{26}$$

$$b_1 = -\frac{3\epsilon_2}{\epsilon_1 + 2\epsilon_2} E_\infty. \tag{27}$$

Solving Eqs.(22,25) with respect to unknowns a_l, b_l (for $l \geq 2$) gives:

$$a_l = b_l = 0, \text{ for } l \geq 2. \tag{28}$$

Thus formulas (18,19) simplify a lot and our final answer for the potential is:

$$\Phi(r < R) = b_1 \, r \cos\theta, \tag{29}$$

$$\Phi(r > R) = \left(\frac{a_1}{r^2} - E_\infty \, r \right) \cos\theta, \tag{30}$$

where a_1 and b_1 are given in terms of physical variables $\epsilon_1, \epsilon_2, E_\infty, R$ in Eqs. (26,27).

Now that I've found the potential $\Phi(r, \theta)$ it is a simple exercise to find the polarized displaced charge on the surface of the sphere:

$$\begin{aligned} \frac{\sigma_{\text{pol}}(\theta)}{\epsilon_0} &= -\left(\frac{\partial \Phi}{\partial r}(R + 0^+, \theta) - \frac{\partial \Phi}{\partial r}(R - 0^+, \theta) \right) = \\ &= \frac{\partial \Phi}{\partial r}(R - 0^+, \theta) \left(1 - \frac{\epsilon_1}{\epsilon_2} \right) = \frac{\epsilon_2 - \epsilon_1}{\epsilon_2} \cos\theta \, b_1, \end{aligned} \tag{31}$$

where in the second line I have used the boundary condition from Eq.(12). Plugging into this equation b_1 from Eq.(27) I get:

$$\sigma_{\text{pol}}(\theta) = -3 \frac{\epsilon_2 - \epsilon_1}{\epsilon_1 + 2\epsilon_2} \, \epsilon_0 E_\infty \cos\theta. \tag{32}$$

Formulas (29,30,32) are main results of these notes (obviously together with formulas for a_1, b_1 in Eqs.(26,27)).

2. NUMERICAL SOLUTION

As I mentioned before I used Nystrom method for numerical solution of the problem. I made the following assumptions. I discritized the surface of the sphere. In my case I choose 400 points. Basically, for each set of nodes we will have number of nodes squared amount of points. This is because we work in spherical coordinate system, so for fixed radius we need to vary only two other coordinates in order to cover the whole surface of the sphere. Thus for one fixed node of a chosen coordinate we will have N coresponding nodes from another coordinate. N is the number of nodes.

The potential inside and outside of the sphere is defined as

$$u, \bar{u} = \mathbf{E}_{\infty} \cdot \mathbf{x} + S\sigma. \tag{33}$$

The boundary conditions are given in the following equation:

$$u_n = R\bar{u}_n, \;\; u = \bar{u}. \tag{34}$$

For the problem that we need to solve, this is known and given relation between potential inside and outside the sphere, where the proportionality constant is called resistivity. It's ratio of conductances of two medium.

$$\bar{u}_n = \mathbf{E}_{\infty} \cdot \hat{n} + \left(D^{\tau} + \frac{1}{2} \right) \sigma, \tag{35}$$

$$u_n = \mathbf{E}_{\infty} \cdot \hat{n} + \left(D^{\tau} - \frac{1}{2} \right) \sigma. \tag{36}$$

From the jump relation we are getting above equations, where D^{τ} is the kernel. I made the following assumption for this problem. I choose two sphere's inner and outer radii. Such that the radius of the inner sphere is slightly smaller than one, while the radius of the outer sphere is slightly bigger than one. I discritized inner and outer spheres and filed kernel with coulomb forces (interacting point charges between any two points on two spheres).

$$\left[(1-R)D^{\tau} - \frac{1+R}{2} \right] \sigma = (R-1)\mathbf{E}_n \cdot \hat{n}. \tag{37}$$

This was final equation that we needed to solve. So solution of this equation revealed me approximateley similar result that was in analitical case.

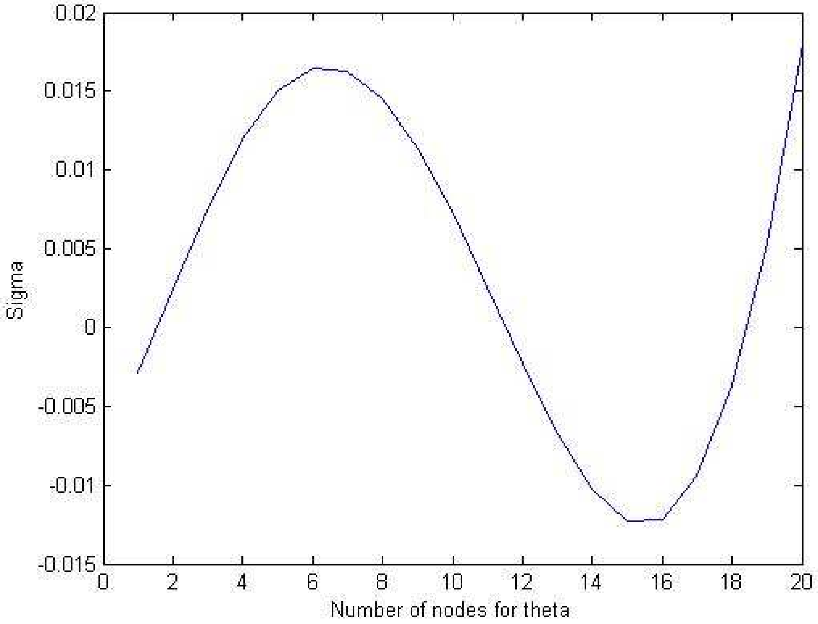

FIG. 1: Numerical solution of the equation 37

3. CONCLUSION

To conclude, the obtained numerical result was not very accurate (ideally it should have been cosine function). I think that the reason of this inaccuracy enters due to several factors. First, I didn't choose enough points. Because Matlab was evaluating for a long time the inverse of the kernel, so for twenty nodes, kernel was a 400 by 400 matrix. Which was already big. Second, the problem that I was solving and the model that I used for comparison were slightly different.

However, numerical method is the only possible way to evaluate potential of the filed and induced distribution of charges for more complicated shapes. Analytical solution is hopeless for arbitrary shapes because we will not have any symmetry.

[1] Etienne Lac and G. M. Homsy, "Axisymmetric deformation and stability of a viscous drop in a steady field" , 2007
[2] http://en.wikipedia.org/wiki/Spherical-harmonics
[3] Lecture/Notes and Homeworks for EM course. Fall 2005 NYU