I worked with Ben and Yong.

Problem 1

By Definition: $Q_{n+1} \perp P_n$

Suppose that Q_{n+1} has 2 identical roots on the interval $[a,b]$, x_1 and x_1.

$$Q_{n+1} = (x - x_1)^2 p \qquad p \in P_{n-1}$$

Let`s choose some number t outside the interval $[a,b]$

$$(x-t)p \in P_n \Rightarrow$$

$$\Rightarrow \int_a^b Q_{n+1}(x-t)p = \int_a^b (x-t)(x-x_1)^2 p^2 \neq 0$$

Problem 2

$$u(s)+\int_{o}^{1} st^3u(t)dt=1, \quad \text{s} \in [\text{a,b}].$$

$$u(s)=1+\alpha s$$

$$1+\alpha s+s\int_0^1 t^3(1+\alpha t)dt=1$$

$$\alpha s+s\int_0^1 t^3dt+s\alpha\int_0^1 t^4dt=0$$

$$\alpha+1/4+\alpha/5=0$$

$$\alpha(1+1/5)=-1/4 \Rightarrow \alpha=-5/24.$$

$$u(s)=1-s*5/24$$

$$\int_0^1 st^3u(t)dt=s\int_0^1 t^3u(t)dt=span(s)\Rightarrow k \text{ is compact.}$$

Problem 3

a)

In composite trapezoid quadrature case the convergence is algebraic, while in Gaussian quadrature case it`s exponential.

For the first scheme error converges to 10^{-5} at N = 250.

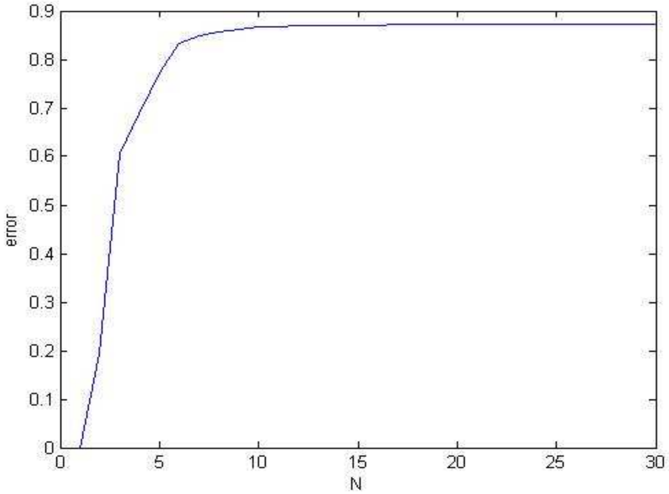
For the second scheme it`s starts when N = 3.

b)

Condition number starts to converge at some N.

Problem 4

a)



b)

It converges, because condition number increases.

Problem 5

$$k(s,t)=k(s-t)$$

$$k(x)=\sum_{m`} a_{m`}\,\mathrm{e}^{im`x}+b_{m`}\,\mathrm{e}^{-im`x}=\sum_{m`} c_{m`}\,\mathrm{e}^{im`x}$$

$Complete\ Set \qquad\qquad m`=0,\pm1,\pm2,.... \qquad m`\in Z.$

$a)$

$$k(s,t)=k(s-t)=\sum_{m`} c_{m`} e^{im`(t-s)} \Rightarrow$$

$$\Rightarrow k\{\mathrm{e}^{imt}\}=\int_0^{2\pi} dt \sum_{m`} c_{m`}\,\mathrm{e}^{im`(t-s)}\,\mathrm{e}^{imt}=$$

$$=\sum_{m`}\int_0^{2\pi} dt c_{m`} e^{im`(t-s)+imt}=$$

$$=\sum_{m`} c_{m`} e^{-im`s}\int_0^{2\pi} dt e^{it(m+m`)}$$

$$if\ \ m`+m\neq 0 \ \Rightarrow \ \int_0^{2\pi} dt e^{it(m`+m)}=\frac{1}{i(m`+m)}(e^{2\pi i(m`+m)}-e^{0})=0$$

$$if\ \ m`+m=0 \ \Rightarrow \ \int_0^{2\pi} dt e^{it(m`+m)}=\int_0^{2\pi} dt=2\pi \ \Rightarrow \ \int_0^{2\pi} dt e^{it(m`+m)}=2\pi\delta_{-m`,m} \ \Rightarrow$$

$$\Rightarrow \ k\{e^{imt}\}=\sum_{m`} 2\pi\delta_{-m`,m} c_{m`} e^{-im`s}=2\pi c_{-m} e^{ims} \ \Rightarrow \ \lambda_m=2\pi c_{-m}$$

e^{imt} *is eigenfunction of* $k(s-t)$ *with eigensolution* λ_m.

$b)$

$$f(s)=\sum_m f_m e^{-ims};\;\; k(x)=\sum_m k_m e^{-imx};\; u(t)=\sum_m u_m e^{-imt}.$$

$$ku=f\;\;\Rightarrow\;\int_0^{2\pi} k(t-s)u(t)dt\equiv f(s)\;\Rightarrow$$

$$\Rightarrow\;\int_0^{2\pi}\left(\sum_{m_1} k_{m_1} e^{-i(t-s)m_1}\right)\left(\sum_{m_2} u_{m_2} e^{-im_2t}\right)dt\equiv\sum_{m_3} f_{m_3} e^{-im_3s}$$

$$\Rightarrow\;\sum_{m_1}\sum_{m_2} k_{m_1}u_{m_2}e^{ism_1}\left(\int_0^{2\pi} e^{-it(m_1+m_2)}dt\right)=\sum_{m_3} f_{m_3}e^{-im_3s}\;\;\Rightarrow$$

$$\Rightarrow\;\sum_{m_1}\sum_{m_2} k_{m_1}u_{m_2}e^{ism_1}\left(2\pi\delta_{m_1,m_2}\right)=\sum_{m_3} f_{m_3}e^{-im_3s}\qquad see\ part\ (a)$$

From completeness of Fourier basis:

$$2\pi k_{-m}u_m=f_m\qquad for\ all\ m\in Z.$$