Transmission Scattering

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Abstract

This paper explores solutions to the Helmholtz equation in a non-homogenous medium. Boundary integral equations are used to find numerical solutions for the scattered wave.

1 Theory

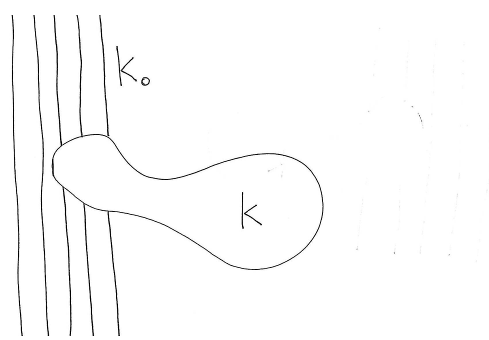
1.1 The Helmholtz Equation

A steady state solution u to the wave equation obeys the Helmholtz equation

$$\left(\Delta + k^2\right) u = 0, \tag{1}$$

where k, the wave number, is a property of the wave-medium. When the wave encounters a scattering obstacle Ω with a non-unitary refractive index n, the effective wave number inside Ω can be related to the original wave number by the equation

$$k \to nk. \tag{2}$$



If we let u represent the solutions outside Ω and v be the solutions inside, then our problem is to find u and v such that

$$\left(\Delta + {k_o}^2\right) u = 0 \tag{3}$$

$$\left(\Delta + k^2\right) v = 0. \tag{4}$$

To do this, we employ the superposition principle, writing u and v as the sums of incident and scattered waves:

$$u = u^{inc} + u^{scat} \tag{5}$$

$$v = v^{inc} + v^{scat} \tag{6}$$

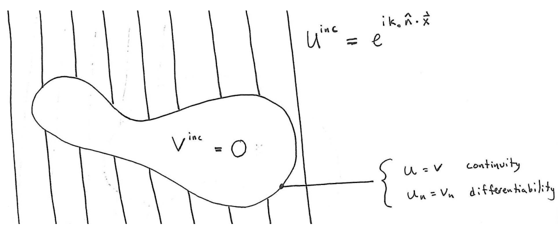
1.2 Boundary Conditions, Incident Waves, and Dirichlet Data

To get a unique solution of physical interest, we impose conditions of continuity and differentiability on the boundary of Ω:

$$u = v \tag{7}$$
$$u_n = v_n \tag{8}$$

where $\frac{\partial u}{\partial n}$ is denoted as u_n. We consider incident waves

$$e^{ik_0 \hat{n}\cdot\vec{x}} = u^{inc} \tag{9}$$
$$0 = v^{inc}. \tag{10}$$



(Note that both (9) and (10) satisfy (1)). Applying (10) and (7) we can define the Dirichleta datum f

$$u^{inc} + u^{scat} = v^{inc} + v^{scat} \tag{11}$$
$$u^{inc} + u^{scat} = v^{scat} \tag{12}$$
$$u^{scat} - v^{scat} = -u^{inc} := f. \tag{13}$$

Differentiating (13) with respect to the normal vector yields the definition for g:

$$u_n{}^{scat} - v_n{}^{scat} = -u_n{}^{inc} := g \tag{14}$$

1.3 Layer Potential Operators and Jump Relations

The fundamental solution to the Helmholtz equation is the zeroth Hankel function of the first kind

$$\Phi\left(\vec{x},\vec{y}\right) = \frac{i}{4} H_0{}^{(1)}\left(k\left|\vec{x}-\vec{y}\right|\right) \tag{15}$$

for x,$y \in \mathbb{R}^2$. At this stage it's useful to model potential fields in terms of the action of an operator. For potential fields a and b, we can write $a = S\sigma$, $b = D\tau$. Taking the limiting values of a and b as their normal derivatives approach $\partial\Omega$ from either side, the following relations can be obtained [1]:

$$a = S\sigma(\vec{x}) \tag{16}$$
$$a_n{}^{\pm}(\vec{x}) = \left(D^T\sigma\right)(\vec{x}) \pm \frac{1}{2}\sigma(\vec{x}) \tag{17}$$
$$b^{\pm}(\vec{x}) = (D\tau)(\vec{x}) \pm \frac{1}{2}\tau(\vec{x}) \tag{18}$$
$$b_n(\vec{x}) = (T\tau)(\vec{x}) \tag{19}$$

where S, D, and T are integral operators from $C(\partial\Omega) \to C(\partial\Omega)$, and we have defined

$$a^{\pm}(\vec{x}) := \lim_{h\to 0} a(\vec{x} \pm h\hat{n}) \tag{20}$$

$$a_n{}^{\pm}(\vec{x}) := \lim_{h\to 0} \left(\hat{n} \cdot \vec{\nabla} a(\vec{x} \pm h\hat{n})\right). \tag{21}$$

Alternatively, we could write these layer operator definitions as improper integrals:

$$(S\sigma)(\vec{x}) := \int_{\partial\Omega} \Phi(\vec{x}, \vec{y})\, \sigma(\vec{y})\, ds_y \tag{22}$$

$$(D\tau)(\vec{x}) := \int_{\partial\Omega} \frac{\partial \Phi(\vec{x}, \vec{y})}{\partial n_y} \tau(\vec{y})\, ds_y \tag{23}$$

$$\left(D^T\sigma\right)(\vec{x}) = \int_{\partial\Omega} \frac{\partial \Phi(\vec{x}, \vec{y})}{\partial n_x} \sigma(\vec{y})\, ds_y \tag{24}$$

$$(T\tau)(\vec{x}) = \int_{\partial\Omega} \frac{\partial^2 \Phi(\vec{x}, \vec{y})}{\partial n_y \partial n_x} \tau(\vec{y})\, ds_y \tag{25}$$

This will ultimately allow us to describe the scattered waves in terms of the action of these operators (which we can determine with a numerical quadrature scheme).

1.4 Single Layer Approach

If we say that

$$u^{scat} = S_o \sigma_o \tag{26}$$

$$v^{scat} = S\sigma \tag{27}$$

(where the subscript indicates that S_o uses wave number k_o) then we can write (13) and (14), our boundary conditions equations, as

$$S_0\sigma_0 - S\sigma = f \tag{28}$$

$$\left(D_0{}^T - \frac{1}{2}\right)\sigma_0 - \left(D^T + \frac{1}{2}\right)\sigma = g \tag{29}$$

or

$$\begin{bmatrix} S_0 & -S \\ D_0{}^T - \frac{1}{2} & D^T + \frac{1}{2} \end{bmatrix} \begin{bmatrix} \sigma_0 \\ \sigma \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix} \tag{30}$$

or

$$\begin{bmatrix} 0 & 0 \\ -\frac{I}{2} & -\frac{I}{2} \end{bmatrix} + \begin{bmatrix} S_0 & -S \\ D_0{}^T & D^T \end{bmatrix} \begin{bmatrix} \sigma_0 \\ \sigma \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix} \tag{31}$$

We would like to right our system in the form of a second kind of Fredholm equation, to take advantage of the uniqueness properties proven for equations of this kind. For an operator P with dirichlet data h this form is

$$(I + P)\, u = h. \tag{32}$$

Unfortunately, (31) is clearly not of this form. Thus another approach is needed.

1.5 Summing Single and Double Layer Potentials

If, instead of the representations used in (26) and (27) we let

$$\sigma_0 = \sigma \tag{33}$$
$$\tau_0 = \tau \tag{34}$$

and build scattered waves with a combination of single and double layer potentials, eg:

$$u^s = D_0\tau + S_0\sigma \tag{35}$$
$$v^s = D\tau + S\sigma \tag{36}$$

then we can write our boundary condition system (13 and 14) as

$$\begin{bmatrix} D_0 + \frac{1}{2} - \left(D - \frac{1}{2}\right) & S_0 - S \\ T_0 - T & {D_0}^T - \frac{1}{2} - \left(D^T + \frac{1}{2}\right) \end{bmatrix} \begin{bmatrix} \tau \\ \sigma \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix} \tag{37}$$

or

$$\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} + \begin{bmatrix} D_0 - D & S_0 - S \\ T_0 - T & {D_0}^T - D^T \end{bmatrix} \begin{bmatrix} \tau \\ \sigma \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}. \tag{38}$$

Negating the second row gives

$$\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} D_0 - D & S_0 - S \\ T - T_0 & D^T - {D_0}^T \end{bmatrix} \begin{bmatrix} \tau \\ \sigma \end{bmatrix} = \begin{bmatrix} f \\ -g \end{bmatrix}. \tag{39}$$

Describing our system in this way has two advantages. First, the single layer potential's singularities have been pitted against one another:

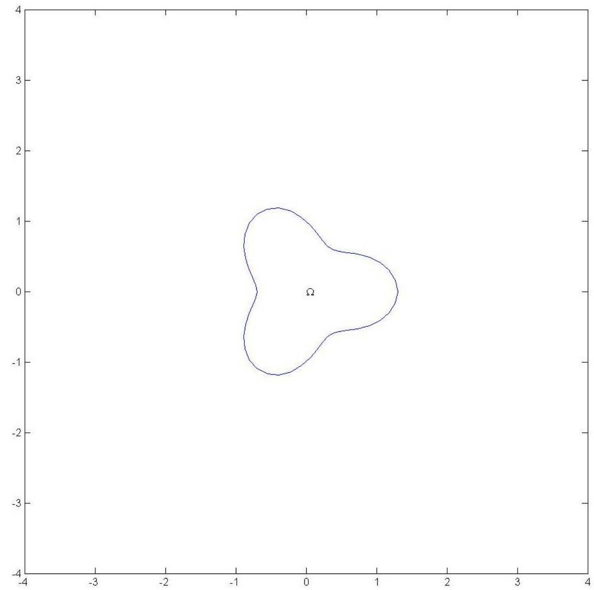
$$S_o - S = \frac{i}{4}{H_0}^1\left(k_0|\vec{x} - \vec{y}|\right) - \frac{i}{4}{H_0}^1\left(k|\vec{x} - \vec{y}|\right). \tag{40}$$

Second, the system is now in the form of a second kind Fredholm equation. An examination of the 2-by-2 block matrix shows that the kernels in equation (39) are continious and weakly singular (Theorem 7.1 in [2]), and thus compact (Theorem 2.22 in [3]).

2 Computation and Numerical Methods

The problem was studied with a scatterer described by the parametric equations

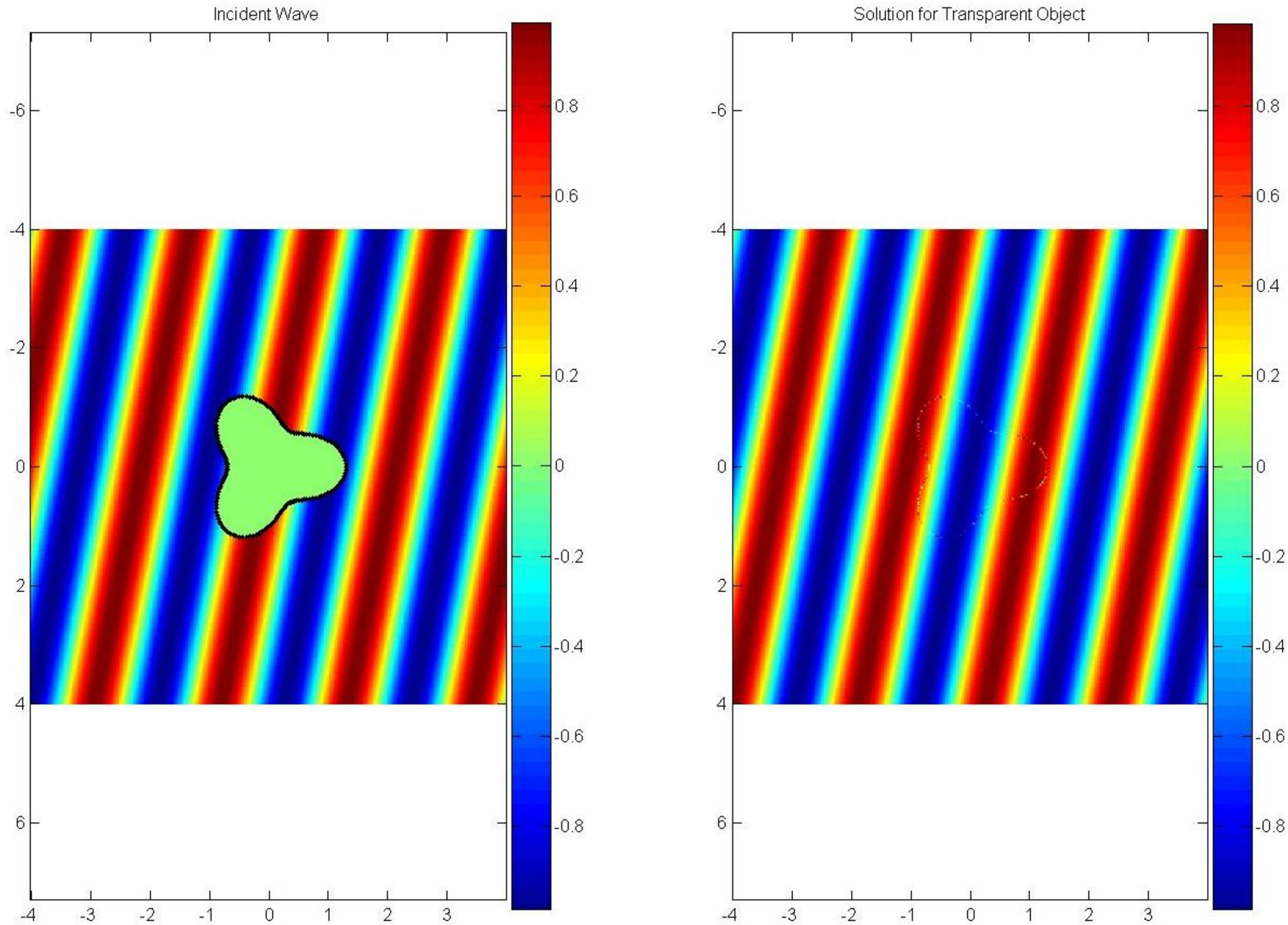
$$x = (1 + .3cos\left(3t\right))\left(cos\left(t\right)\right) \tag{41}$$
$$y = (1 + .3cos\left(3t\right))\left(sin\left(t\right)\right) \tag{42}$$



The kernel matrices used to construct the 2-by-2 block matrix in (39) and the linear solution to this matrix equation were obtained using the Nystrom Method. Thus convergence vs N, the number of quadrature points, was as good as the convergence rate of the chosen quadrature scheme (Kapur-Rokhlin). This quadrature method was chosen for its ability to neutralize problematic entries on the diagonals of the 2-by-2 block matrix in (39). (Specifically, the log singularity of $T - T_o$, and D's lack of analyticity when $\vec{x} = \vec{y} \in \partial\Omega$).

3 Results

For the special case where $k = k_o$, we would expect the wave to pass unimpeded through Ω. This is a somewhat trivial scenario–a wave passing through a transparent obstacle.



Calculated with 150 quadrature points, 267 grid points. k = 3. Convergence Order = 6.

References

[1] Colton, David L., and Rainer Kress. Integral Equation Methods in Scattering Theory. Boston: Krieger Company, 1992.

[2] Rokhlin, V. Solution of Acoustic Scattering Problems by Means of Second Kind Integral Equations. Wave Motion 5 (1983): 257-72.

[3] Kress, Rainer. Linear Integral Equations. New York: Springer, 1990. 19-25.