Math 103 Fall 2009 Midterm Exam

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PRINT NAME: ______________________________

Instructions: This is an open book open notes take home exam. You can also use any printed matter you like. Use of calculators is not permitted. **You must justify all of your answers to receive credit. The exam is due on Monday November 2 at the regular class time. Do all the 10 problems.** Please do all your work on the paper provided.

The Honor Principle requires that you neither give nor receive any aid on this exam.

Grader's use only:

1. ________ /10

2. ________ /10

3. ________ /10

4. ________ /10

5. ________ /10

6. ________ /10

7. ________ /10

8. ________ /10

9. ________ /10

10. ________ /10

Total: ________ /100

1. Let

$$F(x) = \begin{cases} 3x & \text{for } x < 1 \\ 3x+1 & \text{for } x \geq 1 \end{cases}$$

be a function $\mathbb{R} \to \mathbb{R}$. Let μ_F be the corresponding Lebesgue-Stieltjes measure, and let $C \subset [0,1] \subset \mathbb{R}$ be the standard Cantor set. Find $\mu_F(C)$. Prove your answer

2. Let

$$F(x) = \begin{cases} 3x & \text{for } x < 1 \\ 3x+1 & \text{for } x \geq 1 \end{cases}$$

be a function $\mathbb{R} \to \mathbb{R}$ and let μ_F be the corresponding Lebesgue-Stieltjes measure.

Let

$$f(x) = \begin{cases} 0 & \text{for } x \notin \mathbb{Q} \\ e^{\sin x} & \text{for } x \in \mathbb{Q} \end{cases}$$

be a function $\mathbb{R} \to \mathbb{R}$.

Find $\int_{\mathbb{R}} f(x) d\mu_F$. Prove your answer.

3. Let $N \subset [0,1]$ and $C \subset [0,1]$ be the standard Vitali and the Cantor sets. Put

$$F(x) = \begin{cases} 4x+1 & \text{for } x < 0 \\ 1 & \text{for } x \geq 0 \end{cases}$$

Let $\mathcal{M}_F$ and μ_F be the corresponding complete Lebesegue-Stieltjes σ-algebra and measure. Let

$$f(x) = \begin{cases} 17 & \text{for } x \notin N \cap C \\ e^{\cos(x^{2009})} & \text{for } x \in N \cap C \end{cases}$$

Is it true that $f : \mathbb{R} \to \mathbb{R}$ is $(\mathcal{M}_F, B_{\mathbb{R}})$-measurable

4. Let

$$f(x) = \begin{cases} 0 & \text{for } x \leq 0 \\ \frac{1}{(-2)^n} & \text{for } x \in (n, n+1], n = 0, 1, 2, 3, \ldots \end{cases}$$

be a function $\mathbb{R} \to \mathbb{R}$.

Let m be the Lebesgue measure on $\mathbb{R}$. Compute $\int_{\mathbb{R}} f(x)dm$. Prove your answer. Note that f is not a simple function.

5. Let $\mathcal{M}$ be the σ-algebra on $\mathbb{R}$ generated by all possible maps $f : \mathbb{R} \to \mathbb{R}$ such that $f(n) = n^2$ for all $n \in \mathbb{Z}$. (See page 44 of the textbooks for the definitions of a σ-algebra generated by a collection of maps, and note that these maps are not assumed to be continuous.) Describe explicitly all the elements of $\mathcal{M}$. Justify your answer.

6. Let $N \subset [0,1]$ be the Vitali set. For $a \in N$ define the functions $f_a : \mathbb{R} \to \mathbb{R}$ as follows

$$f_a(x) = \begin{cases} -\pi & \text{if } x = a \text{ and } x \leq \frac{1}{2} \\ -e & \text{if } x = a \text{ and } x > \frac{1}{2} \\ 0 & \text{otherwise.} \end{cases}$$

Are the functions $f_a(x), a \in N$ Lebesgue measurable? For $x \in \mathbb{R}$ put $f(x) = \sup_{a \in N} f_a(x)$ and put $g(x) = \inf_{a \in N} f_a(x)$. Are the functions f and g Lebesgue measurable?

7. Let

$$F(x) = \begin{cases} 7 & \text{for } x < -1 \\ 8 + x & \text{for } x \geq -1 \end{cases}$$

be a function $F : \mathbb{R} \to \mathbb{R}$ and let $\mathcal{M}_F$ and μ_F be the corresponding Lebesgue-Stieltjes σ-algebra and measure. Prove or disprove that for every $r, s \in \mathbb{R}$ and for every $E \in \mathcal{M}_F$ both rE and $s + E$ are elements of $\mathcal{M}_F$.

8. Let $\mathcal{A}$ be the algebra of finite and co-finite subsets of $\mathbb{R}$. Let μ_0 be a premeasure on $\mathcal{A}$ defined by

$$\mu_0(E) = \begin{cases} 1 \text{ if } 1 \in E \\ 0 \text{ if } 1 \notin E \end{cases}$$

Let μ^* be the corresponding outer measure. Describe explicitly the σ-algebra of all μ^*-measurable subsets of $\mathbb{R}$. Prove your answer.

9. Let μ be the counting measure on the power set σ-algebra $\mathcal{P}(\mathbb{C})$. Let $f_n : \mathbb{C} \to \mathbb{C}$ be a sequence of functions and let $f : \mathbb{C} \to \mathbb{C}$ be a function. Prove that f_n converges to f in measure if and only if f_n converges to f uniformly.

10. Prove the following statement or construct a counterexample. Let X be a space with $\mu(X) < \infty$ and let $f_n : X \to \mathbb{C}$ be a sequence of measurable functions that converges to a measurable $f : X \to \mathbb{C}$ almost uniformly, then the sequence converges to f in L^1.