Math 103 Fall 2009 Final Exam

Instructor: Vladimir Chernov

Wednesday December 2, 2009

PRINT NAME: ________________________________

Instructions: This is an open book open notes take home exam. You can also use any printed matter you like. Use of calculators is not permitted.

The exam is due by noon on Tuesday December 8. Please deliver the completed exam to my office 304 Kemeny. If the office door is locked, then please slide the exam under the door and write the time you have finished working on the exam. **You must justify all of your answers to receive credit.**

The Honor Principle requires that you neither give nor receive any aid on this exam.

Grader's use only:

1. ________ /10

2. ________ /10

3. ________ /10

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9. ________ /10

10. ________ /10

Total: ________ /100

1. Let $(X, \mathcal{M}, \mu)$ be a space with a positive measure, such that $\mu(X) = 1$. Assume moreover that for every $A \in \mathcal{M}$ with $\mu(A) \neq 0$ there exists $B \in \mathcal{M}$ with $B \subset A$ such that $0 < \mu(B) < \mu(A)$. Either prove that there always has to exist $E \in \mathcal{M}$ with $\mu(E) = \frac{1}{2}$ or construct an example when this is not so. Hint: you may want to use the Zorn's Lemma.

2. Let X, Y be vector spaces over $\mathbb{R}$. Let $||\cdot||_{X,1}$, and $||\cdot||_{X,2}$ be equivalent norms on X. Let $||\cdot||_{Y,1}$ and $||\cdot||_{Y,2}$ be equivalent norms on Y. Let $S : X \to Y$ be a linear operator that is bounded with respect to the norms $||\cdot||_{X,1}$ and $||\cdot||_{Y,1}$. Let $T : Y \to X$ be a linear operator that is bounded with respect to the norms $||\cdot||_{Y,2}$ and $||\cdot||_{X,2}$. Either prove that the composition $T \circ S : (X, ||\cdot||_{X,1}) \to (X, ||\cdot||_{X,1})$ is continuous or find an example when this is false.

3. **a** Prove the following statement or give an example when it is false. Let $(\mathbb{R}^2, ||\cdot||)$ be a two dimensional normed vector space. Then the closed unit ball B centered at the origin $B = \{\vec{x} : ||\vec{x}|| \leq 1\}$ is a convex set.

   b Prove the following statement or give an example when it is false. Let $\mathbb{R}^2$ be the vector space and let B be a convex closed set, then $B = \{\vec{x} : ||\vec{x}|| \leq 1\}$ for some norm $||\cdot||$ on $\mathbb{R}^2$.

4. Let $X = \mathbb{N}$ be the set of positive integer numbers equipped with the power set σ-algebra $\mathcal{P}(\mathbb{N})$. Let μ be the positive measure on $\mathcal{P}(\mathbb{N})$ defined by $\mu(E) = \sum_{e \in E} \frac{1}{e^2}$. Let $g \in L^3(\mu)$ be a function. For $f \in L^3(\mu)$ define $\phi_g(f) = \int_X fg d\mu$. Either prove that $\phi_g : L^3 \to \mathbb{C}$ is a well-defined bounded linear functional, or prove that this is false.

5. Let $\mathcal{L}$ be the Lebesgue σ-algebra on $\mathbb{R}$ and let μ-be the Lebesgue measure. Let ν_1 and ν_2 be the positive measures on $\mathcal{L}$ defined by $d\nu_1 = e^x d\mu$ and $d\nu_2 = e^{-x} d\nu_1$. Compute $\int_{[0,1]} \cos(x) d\nu_2$. Show all the steps.

6. Let $A \subset \mathbb{R}^2$ be the set of all pairs of numbers (x_0, y_0) such that there exists a nonzero finite degree polynomial $P(x, y)$ with **rational** coefficients satisfying $P(x_0, y_0) = 0$. Either prove that A is an element of the Borel σ-algebra $B_{\mathbb{R}^2}$ or show that this is not the case.

7. Let $X = [0,1]$ and let $\mathcal{L}, m$ be the restrictions of the Lebesgue σ-algebra and measure to X. Let $A \subset [0,1]$ be the set of all numbers that are a root of some degree three polynomial with integer coefficients. Is it true that there is a compact set $K \subset \big(([0,1] \times [0,1]) \setminus (A \times A)\big)$ such that $(m \times m)(K) > \frac{1}{2}$? Hint: you might want to use the fact that a closed subset of a compact set is compact and that the product of compact sets is compact.

8. Let $(M, \mathcal{M}, \mu)$ be a complete measure space. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions in L^∞ that converges to f in the normed space L^∞. Let $\{g_n\}_{n=1}^{\infty}$ be a sequence of functions in L^1 that converges to g in the normed space L^1. Prove or disprove that the sequence $f_n g_n$ converges to fg in L^1.

9. Let $X = [0, 1], Y = [0, 1]$ be spaces equipped with the restrictions of the Lebesgue σ-algebra. Let μ and ν be the restrictions of the Lebesgue measure to the spaces X and Y respectively. Let $K \subset [0, 1]$ be the standard Cantor set, and let K^c be its compliment in $[0, 1]$. Prove that the function $f(x, y) = \sin(x+y)\chi_K(x)\chi_{K^c}(y)$ is measurable and integrable over $\big(X \times Y, \mu \times \nu\big)$. Compute $\int_{X\times Y} f(x, y)d(\mu \times \nu)$.

10. Let X be a set and $\mathcal{E} \subset \mathcal{P}(X)$ be a subset of the power set. Prove that the σ-algebra generated by $\mathcal{E}$ is equal to the union of all the σ-algebras generated by $\mathcal{F}$, where $\mathcal{F}$ ranges over all the countable subsets of $\mathcal{E}$. See Exercise 5 page 24.