Math 103 Fall 2007 Midterm Exam

Instructor: Vladimir Chernov

Wednesday, October 24, 2007

PRINT NAME: ______________________________

Instructions: This is an open book open notes take home exam. You can also use any printed matter you like. Use of calculators is not permitted. **You must justify all of your answers to receive credit. The exam is due on Monday October 29 at the regular class time. Do any 9 problems out of 10.** If you want you can do all 10 problems, in this case the problem with the lowest score will be disregarded. Please do all your work on the paper provided. Do not forget to write your name, it costs one point.

The Honor Principle requires that you neither give nor receive any aid on this exam.

Grader's use only:

1. ________ /11

2. ________ /11

3. ________ /11

4. ________ /11

5. ________ /11

6. ________ /11

7. ________ /11

8. ________ /11

9. ________ /11

10. ________ /11

Write your name: ________ /1

Total: ________ /100

1. Let

$$F(x) = \begin{cases} -1 & \text{for } x \leq -1 \\ x & \text{for } x \in (-1, 3) \\ 8 - \frac{3}{x} & \text{for } x \geq 3 \end{cases}$$

be a function $\mathbb{R} \to \mathbb{R}$. Let μ_F be the corresponding Lebesgue-Stieltjes measure. Find $\mu_F([3, +\infty))$. (Note that $[3, +\infty)$ is not an h-interval.) Prove your answer.

2. Let

$$F(x) = \begin{cases} -1 & \text{for } x < 0 \\ 4 & \text{for } x \geq 0 \end{cases}$$

be a function $\mathbb{R} \to \mathbb{R}$. Let μ_F be the corresponding Lebesgue-Stieltjes measure.

Let

$$f(x) = \begin{cases} 0 & \text{for } x \notin \mathbb{Q} \\ 1 & \text{for } x \in \mathbb{Q} \end{cases}$$

be a function $\mathbb{R} \to \mathbb{R}$.

Find $\int_{\mathbb{R}} f(x) d\mu_F$. Prove your answer.

3. Let

$$f(x) = \begin{cases} 0 & \text{for } x \leq 0 \\ \frac{5}{3^n} & \text{for } x \in (n, n+1], n = 0, 1, 2, 3, \ldots \end{cases}$$

be a function $\mathbb{R} \to \mathbb{R}$.

Let m be the Lebesgue measure on $\mathbb{R}$. Compute $\int_{\mathbb{R}} f(x) dm$. Prove your answer. Note that f is not a simple function.

4. Let C be the standard Cantor set. Let

$$f(x) = \begin{cases} e^x & \text{for } x \in C \\ \sin(x) & \text{for } x \notin C \end{cases}$$

be a function $\mathbb{R} \to \mathbb{R}$. Is it true that $f(x)$ is $(\mathcal{L}, B_{\mathbb{R}})$-measurable? Here $\mathcal{L}$ is the standard Lebesgue σ-algebra. Prove your answer.

5. Let $N \subset [0,1]$ be the Vitali set defined in Section 1.1 of the textbook. Let $\overline{\mathcal{M}_F}$ and $\overline{\mu_F}$ be the complete σ-algebra and the Lebesgue- Stieltjes measure corresponding to

$$F(x) = \begin{cases} x & \text{for } x \leq -1 \\ -1 & \text{for } x \in (-1,3) \\ -1+x & \text{for } x \geq 3. \end{cases}$$

Let

$$f(x) = \begin{cases} e^x & \text{for } x \in N \\ 0 & \text{for } x \notin N \end{cases}$$

be a function $\mathbb{R} \to \mathbb{R}$. If $f(x)$ is $(\overline{\mathcal{M}_F}, B_{\mathbb{R}})$-measurable, then compute $\int_{\mathbb{R}} f(x) d\overline{\mu_F}$. If $f(x)$ is not $(\overline{\mathcal{M}_F}, B_{\mathbb{R}})$-measurable, then prove that it is not measurable.

6. Let $N \subset [0,1]$ be the Vitali set defined in Section 1.1 of the textbook. For $a \in N$ define the functions $f_a : \mathbb{R} \to \mathbb{R}$ as follows

$$f_a(x) = \begin{cases} 1 & \text{if } x = a \text{ and } x \leq \frac{1}{2} \\ 3 & \text{if } x = a \text{ and } x > \frac{1}{2} \\ 0 & \text{otherwise.} \end{cases}$$

Are the functions $f_a(x), a \in N$ Lebesgue measurable? For $x \in \mathbb{R}$ put $f(x) = \sup_{a \in N} f_a(x)$ Is this function f Lebesgue measurable? Is this function $(\mathcal{P}(\mathbb{R}), B_{\mathbb{R}})$-measurable? Prove your answers. (Here $\mathcal{P}(\mathbb{R})$ is the power set σ-algebra.)

7. Let $\mathcal{M}$ be the σ-algebra on $\mathbb{R}$ generated by all possible maps $f : \mathbb{R} \to \mathbb{R}$ such that $f(n) = 7$ for all $n \in \mathbb{Z}$. (These maps are not assumed to be continuous). Describe explicitly all the elements of $\mathcal{M}$. Justify your answer.

8. Let $\mathcal{A}$ be the algebra of finite and co-finite subsets of $\mathbb{Z}$. Let μ_0 be a premeasure on $\mathcal{A}$ defined by

$$\mu_0(E) = \begin{cases} 3 \text{ if } 5 \in E \\ 0 \text{ if } 5 \notin E \end{cases}$$

Let μ^* be the corresponding outer measure. Describe explicitly the σ-algebra of all μ^*-measurable subsets of $\mathbb{Z}$. Prove your answer.

9. Let $\mathcal{M}$ be a σ-algebra and let μ be a **finitely** additive measure on $\mathcal{M}$. Assume that for every increasing sequence $\{E_i\}_{i=1}^{\infty}$ of elements of $\mathcal{M}$ we have that $\mu(\cup_{i=1}^{\infty} E_i) = \lim_{i\to\infty} \mu(E_i)$. Prove that μ is a measure. (This problem is one half of Exercise 11 in Section 1.3 of our textbook.)

10. Let $F(x) = x + 3$ be a function $F : \mathbb{R} \to \mathbb{R}$ and let $\mathcal{M}_F$ and μ_F be the corresponding Lebesgue-Stieltjes σ-algebra and measure. Prove or disprove that for every $r, s \in \mathbb{R}$ and for every $E \in \mathcal{M}_F$ both rE and $s + E$ are elements of $\mathcal{M}_F$.