Math 103 Fall 2007 Final Exam

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Friday December 7, 2007

PRINT NAME: ______________________________

Instructions: This is an open book open notes take home exam. You can also use any printed matter you like. Use of calculators is not permitted.

You can start working on the exam in the morning of Friday December 7. **The exam is due by 12:30 PM on Tuesday December 11.** Please deliver the completed exam to my office 304 Kemeny. If the office door is locked, then please slide the exam under the door and write the time you have finished working on the exam. **Do 10 out of 11 problems.** If you want you can do all 11 problems, then your score will be based on the 10 problems that you solved best. **You must justify all of your answers to receive credit.**

The Honor Principle requires that you neither give nor receive any aid on this exam.

Grader's use only:

1. ________ /15

2. ________ /15

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10. ________ /15

11. ________ /15

Total: ________ /150

1. Let $F : \mathbb{R} \to \mathbb{R}$ be a function defined as

$$F(x) = \begin{cases} 3x \text{ for } x < -3 \\ -9 \text{ for } x \in [-3, 4) \\ x - 13 \text{ for } x \geq 4 \end{cases}$$

Let $\mu_F, \mathcal{M}_F$ be the corresponding complete Lebesgue-Stieltjes measure and σ-algebra.

Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded function that vanishes outside of $[-17, 17]$ and let $g = f + \chi_{[-1,1]}(x) \sin x - \chi_{\mathbb{Q}}(x) \cos x$. Prove or disprove that we have $\int_E (g - f) d\mu_F \geq 0$, for every $E \in \mathcal{M}_F$.

2. Let $F : \mathbb{R} \to \mathbb{R}$ be a function defined as

$$F(x) = \begin{cases} -3 \text{ for } x < -3 \\ x \text{ for } x \in [-3, 4) \\ 4 \text{ for } x \geq 4 \end{cases}$$

Let $\mu_F, \mathcal{M}_F$ be the corresponding complete Lebesgue-Stieltjes measure and σ-algebra. Let $g \in L^{\mathbf{5}}(\mu_F)$ be a function. For $f \in L^2(\mu_F)$ put $\phi_g(f) = \int fg d\mu_F$. Prove that $\phi_g \in (L^2(\mu_F))^*$.

3. Let X be a vector space equipped with two equivalent norms $||\cdot||_{X1}$ and $||\cdot||_{X2}$. Let Y be a vector space equipped with two equivalent norms $||\cdot||_{Y1}$ and $||\cdot||_{Y2}$. Let $T: X \to Y$ be a linear functional, that is bounded in the sense of $||\cdot||_{X1}$ and $||\cdot||_{Y1}$ norms. Prove that T is a bounded linear functional in the sense of $||\cdot||_{X2}$ and $||\cdot||_{Y2}$ norms.

4. Let $X = [0, 2\pi]$ and let $\mathcal{L}$ and m be the restrictions of the Lebesgue σ-algebra and measure to X. Let ν be the signed measure defined by $d\nu = \sin x dm$. Compute $\int_X \cos x d\nu$. Justify all your steps.

5. Let $(X, \mathcal{M}, \mu)$ be a complete measure space. Let $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ be sequences of measurable functions in $L^1(\mu)$. Assume that $f, g \in L^1(\mu)$ are such that $f_n \to f$ and $g_n \to g$ in $L^1(\mu)$. Prove or disprove that $f_n - g_n \to f - g$ in measure.

6. Let $X = [0,1], Y = [0,5]$ be spaces equipped with the restrictions of the Lebesgue σ-algebra. Let μ and ν be the restrictions of the Lebesgue measure to the spaces X and Y respectively. Let $K \subset [0,1]$ be the standard Cantor set, and let K^c be its compliment in $[0,5]$.

   Prove that the function $f(x,y) = \sin(xy)\chi_K(x)\chi_{K^c}(y)$ is measurable and integrable over $\big(X \times Y, \mu \times \nu\big)$. Compute $\int_X \int_Y f(x,y) d\nu d\mu$.

7. Let $(X, ||\cdot||)$ be a Banach space, and let $V \subset X$ be a proper closed subspace of X Put $Y = X/V$ to be the quotient space. Show that $||x+V|| = \inf_{y\in V} ||x+y||$ is a norm on the quotient space and that it is well defined, i.e. if $x_1 + V = x_2 + V$ then $||x_1 + V|| = ||x_2 + V||$. See Exercise 12, Section 5.1.

8. Let $X = [0,1]$ and let $\mathcal{L}, m$ be the restrictions of the Lebesgue σ-algebra and measure to X. Given $\epsilon > 0$ explain how to construct a compact $K \subset \big((X \times X) \setminus (\mathbb{Q} \times \mathbb{Q})\big)$ such that $(m \times m)(K) > 1 - \epsilon$. Hint: you might want to use the fact that a closed subset of a compact set is compact and that the product of compact sets is compact.

9. Let X be a set and $\mathcal{E} \subset \mathcal{P}(X)$ be a subset of the power set. Prove that the σ-algebra generated by $\mathcal{E}$ is equal to the union of all the σ-algebras generated by $\mathcal{F}$, where $\mathcal{F}$ ranges over all the countable subsets of $\mathcal{E}$. See Exercise 5 page 24.

10. Let $X = [0,1]$ and let $\mathcal{L}, m$ be the restrictions of the Lebesgue σ-algebra and measure to X. Define the signed measure ν on $\mathcal{L}$ by

$$\nu(E) = m(E) + \sum_{n \text{ positive integer such that } \frac{1}{n} \in E} \frac{(-1)^n}{n^3}.$$

Describe explicitly the Jordan Decomposition of the signed measure ν. Prove your answer.

11. Prove that the complex conjugation function $\phi : \mathbb{C} \to \mathbb{C}$ defined by $\phi(z) = \overline{z}$ is a $(B_{\mathbb{C}}, B_{\mathbb{C}})$-measurable function. Here $B_{\mathbb{C}}$ is the Borel σ-algebra for $\mathbb{C}$.