Math 103 Fall 2005

Topics in Analysis

Midterm Exam

Thursday October 27, 2005

The exam is due at 11:50 PM Thursday November 3, 2005 in the Instructor's office 414 Bradley Hall.

Your name (please print): ______________________________

Instructions: This is an open book open notes exam. You can consult any printed matter you like, but **you can not consult other humans. Use of calculators is not permitted**. **You must justify all of your answers to receive credit,** unless instructed otherwise. If I am not in my office when you submit your exam, then please write the time you finished working on it and slide it under my office door.

The exam total score is the sum of the 10 best (out of 11) problem scores. Please do all your work on the paper provided.

The Honor Principle requires that you neither give nor receive any aid on this exam.

Grader's use only

1. ________ /10
2. ________ /10
3. ________ /10
4. ________ /10
5. ________ /10
6. ________ /10
7. ________ /10
8. ________ /10
9. ________ /10
10. ________ /10
11. ________ /10

Total: ________ /100

(1) Let $f : \mathbb{C} \to \mathbb{R}$ be a function such that $f(z) = |z|$ if $|z| \neq 1$; and $f(z) = 2$ if $|z| = 1$. Is the function $(B_{\mathbb{C}}, B_{\mathbb{R}})$-measurable? (Explain your answer.)

(2) Let $\mathcal{M}$ be a σ-algebra on $\mathbb{R}$ of countable and cocountable sets. Let μ be a measure on $\mathcal{M}$ defined by $\mu(E) = 0$ if E is countable and $\mu(E) = 1$ if E is cocountable. Let N be the set defined on page 20 of the textbook, that is not measurable with respect to the Lebesgue measure. Is N an element of $\mathcal{M}$? If it is, what is its measure? (Explain your answer.)

(3) Let $f, g : \mathbb{R} \to \mathbb{R}$ be measurable functions such that $g(x) \neq 0$ for all $x \in \mathbb{R}$. Prove that $\dfrac{f}{g}$ is a measurable function.

(4) Let $\mathcal{M}$ be the trivial σ-algebra on $\mathbb{R}^1$, i.e. the σ-algebra whose only elements are $\emptyset$ and $\mathbb{R}^1$. Is $[0,1] \times [0,1] \subset \mathbb{R}^2$ measurable with respect to the $\mathcal{M} \otimes B_{\mathbb{R}^1}$ σ-algebra on $\mathbb{R} \times \mathbb{R}$?

(5) Prove that if $f : \mathbb{R} \to \mathbb{R}$ is monotone, then f is Borel measurable. (Exercise 8, Section 2.1)

(6) Let $E_i \in B_{\mathbb{R}}, i = 1, 2$ be two disjoint subsets of $\mathbb{R}$. Assume that $f : \mathbb{R} \to \mathbb{R}$ is a function such that $f|_{E_i}, i = 1, 2$ are measurable (with respect to the σ-algebras coming from $B_{\mathbb{R}^1}$, see page 44). Prove that if $E_1 \sqcup E_2 = \mathbb{R}$, then $f : \mathbb{R} \to \mathbb{R}$ is Borel measurable.

(7) Let $f_n(x) : [0,1] \to \mathbb{R}, n = 1, \cdots, \infty$ be functions defined as

$$f_n(x) = \begin{cases} 5 - \frac{1}{n} e^x \cos(nx) & \text{if } x \notin \mathbb{Q} \cap [0,1] \\ 0 & \text{if } x \in \mathbb{Q} \cap [0,1] \end{cases}$$

Find $\lim_{n\to\infty} \int_{[0,1]} f_n dm$. (Prove your answer.)

(8) Let $f \in L^+$ be a function $\mathbb{R} \to \mathbb{R}$. Prove that $\lim_{n\to\infty} \int_{[-n,n]} f dm = \int_{\mathbb{R}} f dm$.

(9) Let X be a space and $\mathcal{A}$ be the trivial algebra consisting of $\emptyset$ and X. Let $\mu_0 : \mathcal{A} \to [0, +\infty]$ be a premeasure defined by $\mu_0(\emptyset) = 0$ and $\mu_0(X) = 1$. List all the subsets of X that are μ^* measurable, where μ^* is an outer measure induced by μ_0. (Prove your answer.)

(10) Let $C \subset [0,1]$ be the Cantor set defined on page 38. Let $f : [0,1] \to \mathbb{R}$ be a function defined as follows

$$f(x) = \begin{cases} 1 & \text{if } x \notin C \\ x & \text{if } x \in C \end{cases}$$

Find $\int_{[0,1]} f dm$. (Prove your answer.)

(11) $f : \mathbb{R} \to \mathbb{R}$ is an L^+-function and $\int f dm = 3$. Prove that there exists $E \in B_{\mathbb{R}}$ of **finite Lebesgue measure** such that $\int_E f dm > 2.5$.