Math 103 Fall 2005

Topics in Analysis

Midterm Exam

Thursday December 1, 2005

The exam is due at 11:50 PM Wednesday December 7, 2005 in the Instructor's office 414 Bradley Hall.

Your name (please print): ____

Instructions: This is an open book open notes exam. You can consult any printed matter you like, but **you can not consult other humans. Use of calculators is not permitted**. **You must justify all of your answers to receive credit,** unless instructed otherwise. If I am not in my office when you submit your exam, then please write the time you finished working on it and slide it under my office door.

The exam total score is the sum of the 10 best (out of 11) problem scores. Please do all your work on the paper provided.

The Honor Principle requires that you neither give nor receive any aid on this exam.

Grader's use only

1. ________ /15

2. ________ /15

3. ________ /15

4. ________ /15

5. ________ /15

6. ________ /15

7. ________ /15

8. ________ /15

9. ________ /15

10. ________ /15

11. ________ /15

Total: ________ /150

(1) (Exercise 4, page 88) ν is a signed measure on $(X, \mathcal{M})$, and ν^+, ν^- are positive measures such that $\nu = \nu^+ - \nu^-$ and $\nu^+ \perp \nu^-$ (i.e. they are mutually singular). Let λ, μ be two positive measures such that $\nu = \lambda - \mu$. Show that $\lambda \geq \nu^+$ and $\mu \geq \nu^-$. (Here $\lambda \geq \nu^+$ means that $\lambda(E) \geq \nu^+(E)$, for every $E \in \mathcal{M}$.)

(2) Let $\mathcal{X}$ be a Banach space, let $L(\mathcal{X}, \mathcal{X})$ be the space of all bounded linear operators $\mathcal{X} \to \mathcal{X}$. Let $S \subset L(\mathcal{X}, \mathcal{X})$ be the subset of all **non**invertible operators. (See page 154 for the definition of invertible operators.)

a: Show that S is a closed subset of $L(\mathcal{X}, \mathcal{X})$.

b: Give an example that shows that S is not necessarily a vector subspace of $L(\mathcal{X}, \mathcal{X})$.

(3) Consider the Banach space $(\mathbb{R}^n, ||\cdot||)$ where for $\vec{x} = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ norm $||\vec{x}|| = \sqrt{(x_1^2 + x_2^2 + \cdots + x_n^2)}$. (You do not have to show that this is a norm.) Let $T : (\mathbb{R}^n, ||\cdot||) \to (\mathbb{R}^n, ||\cdot||)$ be a linear operator. Show that T is bounded.

(4) Consider $\mathbb{Z}$ equipped with a power set σ-algebra $\mathcal{P}(\mathbb{Z})$. Let μ be a measure on $\big(\mathbb{Z}, \mathcal{P}(\mathbb{Z})\big)$ such that $\mu(\{n\}) = e^{|n|}$ for $n \in \mathbb{Z}$. Let $f : \mathbb{Z} \to \mathbb{R}$ be a positive $\mathcal{P}(\mathbb{Z})$-measurable function such that $\int f d\mu < \infty$, show that $\int f^2 d\mu < \infty$.

(5) Let $\mathcal{E} = \{(q_1, q_2) \subset \mathbb{R} | q_1, q_2 \in \mathbb{Q}\}$. Prove that $\mathcal{M}(\mathcal{E}) = B_{\mathbb{R}^1}$.

(6) Let $N \subset [0,1]$ be the Lebesgue non-measurable set defined on page 20 and let $C \subset [0,1]$ be the Cantor set defined on page 38. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined as

$$f(x) = \begin{cases} e^x & \text{if } x \notin (C \cap N) \\ x & \text{if } x \in C \cap N \end{cases}$$

Prove that f is Lebesgue measurable.

(7) Let f be a function defined as

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in [-1,1] \setminus \mathbb{Q} \\ \frac{1}{x^2} & \text{if } x \in \mathbb{R} \setminus \big(\mathbb{Q} \cup [-1,1]\big) \end{cases}$$

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Compute $\int_{\mathbb{R}} f dm$, where m is the Lebesgue measure. Explain your steps. If you wish to use Theorem 2.28 be extra careful.

(8) Let $C \subset [0,1]$ be the Cantor set defined on page 38 and let $f(x,y)$ be a function $[0,1] \times [0,1] \to \mathbb{R}$ defined as

$$f(x,y) = \begin{cases} e^{x^2 y} & \text{if } x \in C \\ \sin(\frac{3\pi x}{2})y & \text{if } x \notin C \end{cases}$$

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Compute $\int_{[0,1]\times[0,1]} f d(m \times m)$. Here m is the Lebesgue measure (or better to say the restriction of the Lebesgue measure to $[0,1]$).

(9) Assume that $f_n \to g$ in L^1 and that $f_n^3 \to h$ in L^1. Prove that $g^3 = h$ almost everywhere.

(10) Assume that f, g are functions in L^1 such that $||f+g||_{L^1} = ||f||_{L^1} + ||g||_{L^1}$. Prove that f and g have the same sign almost everywhere.

(11) Let $f(x) : [0,1] \to R$ be defined as

$$f(x) = \begin{cases} x^2 \text{ if } x \notin \mathbb{Q} \\ x^3 \text{ if } x \in \mathbb{Q} \end{cases}$$

Given $\epsilon > 0$ **construct an open set** $E \subset [0,1]$ with $m(E) < \epsilon$ such that $f\big|_{E^c}$ is continuous.