Math 101. *Topics in Algebra.*

Homework 5. Due on Friday, 10/31/2008.

1. Let G be a group of order 99. Show that either $G \cong \mathbb{Z}_{99}$ or $G \cong \mathbb{Z}_3 \times \mathbb{Z}_{33}$.

2. Denote by $Aut(\mathbb{Z}_n)$ the group of automorphisms of $\mathbb{Z}_n$ (viewing $\mathbb{Z}_n$ as an additive group). Show that $Aut(\mathbb{Z}_n) \cong \mathbb{Z}_n^\times$, where $\mathbb{Z}_n^\times$ denotes the multiplicative group of the ring $\mathbb{Z}_n$.

3. (a) Suppose that H_1, H_2 and K are groups, $\sigma : H_1 \to H_2$ is an isomorphism, and $\psi : H_2 \to Aut(K)$ a homomorphism, so that $\varphi = \phi \circ \sigma : H1 \to Aut(K)$ is also a homomorphism. Show that $K \rtimes_\varphi H_1 \cong K \rtimes_\varphi H_2$.

   (b) Suppose that H and K are groups and $\varphi, \psi : H \to Aut(K)$ are monomorphisms with the same image in $Aut(K)$. Show that there exists a $\sigma \in Aut(H)$ such that $\psi = \varphi \circ \sigma$.

   (c) Suppose that H and K are groups, $\varphi, \psi : H \to Aut(K)$ are monomorphisms, and $Aut(K)$ is finite and cyclic. Show that φ and ψ have the same image in $Aut(K)$.

   (d) Let $p < q$ be primes with $p|(q-1)$. Let H and K be cyclic groups of order p and q respectively. Let $\varphi, \psi : H \to Aut(K)$ be nontrivial homomorphisms. Show that $K \rtimes_\varphi H \cong K \rtimes_\psi H$.

4. Let $p < q$ be primes, and let G be a group of order pq. We know from class that $G \cong \mathbb{Z}_q \rtimes_\varphi \mathbb{Z}_p$ for some $\varphi : \mathbb{Z}_p \to Aut(\mathbb{Z}_q)$. By analyzing all possible φ, find (up to isomorphism) all groups of order pq. If $p = 2$, describe them without using semidirect products.