Math 101. *Topics in Algebra.*

Homework 1 (warm-up). Due on Wednesday, 10/1/2008.

1. Without quoting a formula from the book, show that if F is a finite field with q elements, then the group $GL_n(F)$ (assume $n \geq 1$) has cardinality $(q^n - 1)(q^n - q) \ldots (q^n - q^{n-1})$. *Hint:* You may use without proof that an $n \times n$ matrix is invertible if and only if its rows (or columns) are linearly independent.

2. (Proof or counterexample.) Let G be a finite group of order n. Then G has at most finitely many isomorphism types, that is there is a finite set of groups, so that G is necessarily isomorphic to one in the finite set. If you decide the number is finite, can you give an upper bound in terms of n?

3. (Proof or counterexample.) Consider the validity of a cancelation law for direct products: Let G, H, K be groups and suppose that $G \times H \cong G \times K$. Then $H \cong K$.

4. Let $G = \{\zeta \in \mathbb{C} \,|\, \zeta^n = 1 \text{ for some } n \geq 1\}$, that is, the set of roots of unity in $\mathbb{C}$. Verify that G is a group. Let $k > 1$ be a fixed integer and consider the map $\zeta \mapsto \zeta^k$. Show that this map is a surjective homomorphism from G to G, but not necessarily an isomorphism.

5. Show that if $n \geq m$ then the number of m-cycles in S_n is given by

$$\frac{n(n-1)(n-2)\ldots(n-m+1)}{m}.$$

6. What is the order of the element of S_9 with cycle decomposition (1634)(287)(59)?