Various Geometric Demonstrations[1]

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Abstract

In this article, our most noted author not only demonstrates a certain theorem proposed by Fermat to be proved by geometers, but also some other theorems concerning the nature of the quadrilateral, which we know to have been demonstrated by some other lovers of geometry, to whom it was proposed by our author.

Our author admits that, at first glance, these theorems appear to involve little difficulty and that their truth is easily understood through analysis. But it is another matter altogether, if [the proofs of these theorems] are to be understood by those who are not trained in the art of analysis. To this end the aforementioned Fermat asked for a geometrical demonstration, which should be prepared in the fashion of the geometers of old, and which could also be understood by those who are not accustomed to analysis. Having addressed the matter, the author has provided purely geometrical demonstrations of all these theorems, in which there is no trace of analysis, and which have been drawn up in such a way that they may be reviewed here conveniently and read by practitioners of geometry in this very article.

1. A certain proposition is found in Fermat's collected letters, which he proposed to be demonstrated by geometers. Although this proposition concerns the nature of a circle and at first glance does not seem difficult at all, yet it has been attempted unsuccessfully by many geometers and, up to now, no demonstration of it has ever been provided. By analysis, its truth is indeed perceived easily, and it is not very difficult to derive a demonstration, but demonstrations of that kind for the most part so smack of analysis that they can scarcely be understood except by experts in this art. Therefore, a geometrical demonstration of this proposition is sought, one prepared in the fashion of the geometers of old, which may also be understood by those who are not accustomed to analysis. I, therefore, will provide such a demonstration here; it begins with the following lemma.

[1] English translation prepared in 2005 by Adam Glover, student, Georgetown College, under the supervision of Homer White. The translation is based on the version in the *Opera Omnia,* Series I Volume 26, edited by Andreas Speiser.

Lemma

2. If a straight line AB (Fig. 1) is cut arbitrarily at two points R and S,

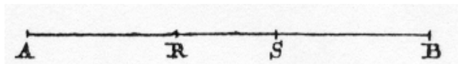


Fig. 1

[then] the rectangle formed from the whole line AB and the middle part RS, together with the rectangle from the outer parts AR and BS, will be equal to the rectangle whose sides are the parts AS and BR; that is,

$$AB \cdot RS + AR \cdot BS = AS \cdot BR.$$

Demonstration

Since

$$AB = AS + BS$$

by multiplying RS on both sides we get:

$$AB \cdot RS = AS \cdot RS + BS \cdot RS$$

Add $AR \cdot BS$ on both sides and we will have:

$$AB \cdot RS + AR \cdot BS = AS \cdot RS + BS \cdot RS + AR \cdot BS$$

But

$$BS \cdot RS + AR \cdot BS = BS\,(RS + AR) = BS \cdot AS$$

Therefore

$$AB \cdot RS + AR \cdot BS = AS \cdot RS + BS \cdot AS$$

But

$$AS \cdot RS + BS \cdot AS = AS\,(RS + BS) = AS \cdot BR$$

Consequently, we will have:

$$AB \cdot RS + AR \cdot BS = AS \cdot BR$$

Q.E.D.

Scholion

3. This lemma can also be demonstrated in the following way by a geometrical figure alone (Fig. 2).

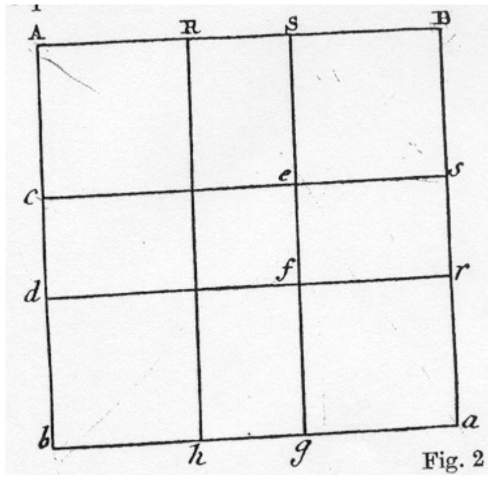


Fig. 2

Upon the given straight line AB divided at points R and S, let there be erected a square $ABab$ and, in a similar way, let side Ba be intersected at points r and s, such that: $Bs = BS$, $sr = SR$, $ar = AR$; then, after drawing straight lines Rh and Sg (and similarly lines sc and rd) parallel to the sides of the square, we will get square parts Ss and cg located along the diagonal Bb. Thus we will have[2]:

$$\square\, AE = \square\, ae$$

Add the rectangle cf on both sides and it becomes:

$$\square\, Ae + \square\, cf = \square\, ae + \square\, cf$$

Or

$$\square\, Af = \square\, ae + \square\, cf;$$

But

$$\square\, ae = \square\, af + \square\, er;$$

Whence

[2](Translator's Note) The $\square$ symbol apprently denotes the area of a recatngle having a given diagonal.

$$\begin{aligned}\square\, Af &= \square\, af + \square\, er + \square\, cf \\ &= \square\, af + \square\, cr\end{aligned}$$

Yet

$$\square\, Af = BS \cdot Br = AS \cdot BR$$

And

$$\square\, af = ar \cdot BS = AR \cdot BS \text{ and } \square\, cr = AR \cdot BS = AS \cdot BR,$$

By substituting the values, we get:

$$AS \cdot BR = AR \cdot BS + AB \cdot RS, \text{ or } AB \cdot RS + AR \cdot BS = AS \cdot BR,$$

We will use this lemma straight ahead.

Fermat's Theorem

4. If upon the diameter AB of the semicircle AMB (Fig. 3) there is a parallelogram $ABFE$, whose latitude AE or BF is equal to the chord of a quarter of the same circle or to the side of the inscribed square, and from points E and F there are two lines EM and FM drawn to any point M on the periphery, then the diameter AB will be cut at points R and S, such that: $AS^2 + BR^2 = AB^2$.

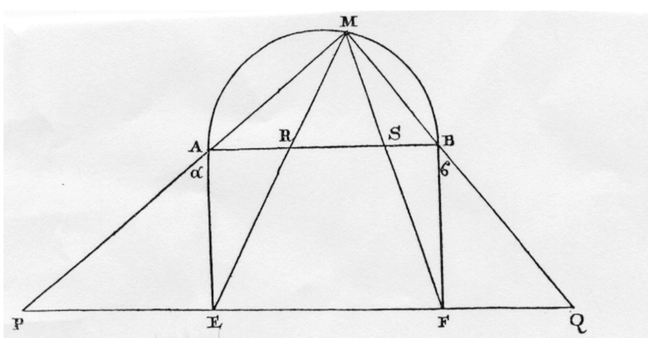


Fig. 3

Demonstration

From point M, let straight lines MAP and MBQ be drawn through the ends A and B of the diameter, until the bases EF meet at points P and Q. Now, since the angle AMB is right, $\angle P + \angle Q$ will be a right angle; but $P + \alpha$ is also a right angle, as is $Q + \beta$, since the lines AE and BF are perpendicular to EF. Whence $\angle P = \angle\beta$ and $\angle Q = \angle\alpha$ and, therefore, triangles PEA and BFQ are similar.

Thus we have, $PE : AE = BF : QF$, and therefore $PE \cdot QF = AE \cdot BF = AE^2$ and for that reason $2PE \cdot QF = 2AE^2$. And because AE is equal to the chord of the square [inscribed in the circle], we will have $2AE^2 = AB^2 = EF^2$, in such a way that $2PE \cdot QF = EF^2$. Wherefore since here the line PQ is cut in such a way at points E and F that two times the product of the outer parts PE and QF is equal to the square of the middle part EF—and indeed with the diameter AB cut in a similar way at points R and S—it also follows that two times the product of the outer parts AR and BS will be equal to the square of the inner part RS; that is, $2AR \cdot BS = RS^2$. Now since $AS + AR = AB + RS$, by taking the squares we get:

$$AS^2 + BR^2 + 2AS \cdot BR = AB^2 + RS^2 + 2AB \cdot RS.$$

Here let RS^2 be replaced by its value $2AR \cdot BR$ and we get:

$$AS^2 + BR^2 + 2AS \cdot BR + AB^2 + 2AB \cdot RS + 2AR \cdot BS.$$

And by the lemma mentioned above, it is the case that

$$AB \cdot RS + AR \cdot BS = AS \cdot BR$$

and therefore also that

$$2AB \cdot RS + 2AR \cdot BS = 2AS \cdot BR$$

After having substituted this value into the equality, we have:

$$AS^2 + BR^2 + 2AS \cdot BR = AB^2 + 2AS \cdot BR;$$

Finally let the common part $2AS \cdot BR$ be removed from both sides and we get:

$$AS^2 + BR^2 = AB^2.$$

Q.E.D.

5. There is a common rule[3] for finding the area of a triangle given its three sides. According to this rule, the single sides should be subtracted separately from the semiperimeter and the "solid," that is, the product, of these three remaining sides should be multiplied by the semiperimeter itself. Then the square root of this product should be taken, which will yield the area of the proposed triangle. Analytically, this rule is indeed demonstrated easily and appropriately prepared demonstrations from analysis appear everywhere, but these differ significantly from the geometric method [of proving theorems], such that they cannot be understood except by those who are practiced in analysis. For this reason I will provide here a purely geometrical demonstration of this rule, in which there is no trace of analysis. The rule from a circle inscribed in a

[3] (Translator's Note) Euler will now describe and prove Heron's Formula for the area of a triangle.

triangle has been sought. This [proof] is approached by way of a circle inscribed in a triangle, whose properties have been sufficiently explained by Euclid; I will address these [properties] in the following propositions, for which I have also the task of forming a demonstration; this demonstration will prepare the way for the demonstration of the aforementioned rule.

Theorem

6. The area of any triangle ABC (Fig. 4) is equal to the product of half the sum of the sides and the radius of the inscribed circle; that is,

$$Area \triangle ABC \text{ is } \frac{1}{2}(AB + AC + BC)OP.$$

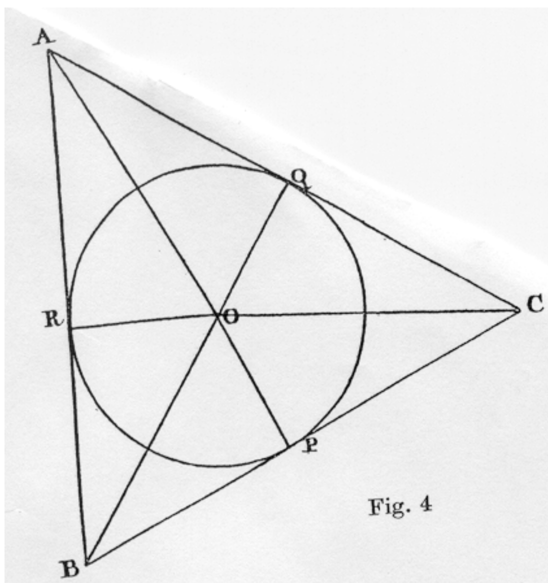


Fig. 4

Demonstration

From the center O of the inscribed circle, let perpendicular lines OP, OQ, and OR be drawn to each of the sides. These lines will be equal to the radius of the inscribed circle. From O let lines OA, OB, and OC be drawn to the angles. These will divide the proposed triangle into three triangles AOB, AOC, BOC, each having the same altitude $OR = OQ = OP$, and whose bases are sides AB, AC, and BC of the triangle. From here, these triangles summed together are equal to the triangle, whose base is the sum of the sides $AB + AC + BC$ and whose altitude is equal to the radius OP of the inscribed circle, to which consequently the area of the proposed triangle ABC itself is equal. This area is equal to the product from half the sum of the sides and the radius OP of the inscribed circle; that is,

$$Area \triangle ABC = \frac{1}{2}(AB + AC + BC)OP.$$

Q.E.D.

Theorem

7. If we draw perpendicular lines OP, OQ, and OR from the center O of the circle inscribed in the triangle ABC, the sides will be cut by these lines in such a way that the posited semiperimeter $\frac{1}{2}(AB + AC + BC) = S$ will be

$$AR = AQ = S - BC, \; BR = BP = S - AC, \text{ and } CP = CQ = S - AB,$$

and

$$AR + BP + CQ = S.$$

Demonstration

But now because the perpendiculars OP, OQ, and OR are equal, it is at once evident that

$$AQ = AR, \; BP = BR, \; and \; CP = CQ,$$

Whence the sum of the sides will be

$$AB + AC + BC = 2AR + 2BP + 2CQ,$$

and for that reason we have:

$$AB + AC + BC = sum \; of \; the \; sides = S.$$

We have, then,

$$\begin{aligned} AR + BC &= S, \text{ and therefore, } AR = AQ = S - BC \\ BP + AC &= S, \text{ and therefore, } BP = BR = S - AC \\ CQ + AB &= S, \text{ and therefore, } CQ = CP = S - AB \end{aligned}$$

Q.E.D.

Theorem

8. If, as before (Fig. 5), we let perpendicular lines OP, OQ, and OR be drawn from the center O of the circle inscribed in triangle ABC to each side, we get the product contained under the parts $AR \cdot BP \cdot CQ$ equal to the product of S—which is half the sum of sides—and the square of the radius OP of the inscribed circle; that is,

$$AR \cdot BP \cdot CQ = S \cdot OP^2.$$

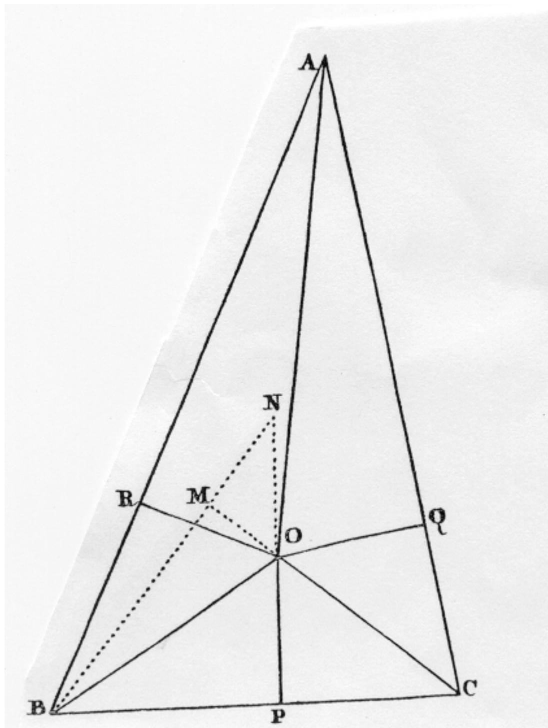


Fig. 5

Demonstration

Draw straight lines OA, OB, and OC from the center O of the inscribed circle to the angles. Now to one of these lines CO (extended, if necessary) let there be drawn a perpendicular line BM from one of the other remaining angles B. This line will meet the radius PO at point N. Now since angles A, B, and C are bisected by the lines OA, OB, and OC, the external angle BOM of triangle BOC will equal $\frac{1}{2}B + \frac{1}{2}C$. From here, since $BOM + OBM$ equals a right angle, we will have:

$$\frac{1}{2}B + \frac{1}{2}C + OBM = \text{right angle.}$$

Now since $A + B + C$ equals two right angles, we will also have

$$\frac{1}{2}B + \frac{1}{2}C + \frac{1}{2}A = \text{right angle and therefore } \frac{1}{2}B + \frac{1}{2}C + OBM = \frac{1}{2}B + \frac{1}{2}C + A,$$

Whence $OBM = \frac{1}{2}A = OAR$. Therefore, since in right triangles BOM and AOR *angle* $OBM =$ *angle* OAR, these [triangles] are similar. As a result we get:

$$AR : RO = BM : MO, \text{ that is, } AR : OP = BM : MO.$$

Furthermore, since triangles CBM, NBP, and NOM are similar, we will get:

$$BM : BC = MO : ON, \text{ that is, } BM : MO = BC : ON.$$

Whence we gather that $AR : OP = BC : ON$, and with the area of the middle and extreme parts equal, it will be: $AR \cdot ON = OP \cdot BC$. Furthermore, since

$$ON = PN - OP : AR \cdot PN - AR \cdot OP = BC \cdot OP$$

That is,

$$AR \cdot PN = AR \cdot OP + BC \cdot OP = (AR + BC)OP.$$

Indeed $AR + BC = S$ (from the preceding paragraph), such that $AR \cdot PN = S \cdot OP$. Finally, since triangles COP and NBP are similar, it is the case that $PN : BP = CP : OP$. Whence $OP \cdot PN = BP \cdot CP$ and $AR \cdot BP \cdot CP = AR \cdot OP \cdot PN$. But the previous equation, when multiplied by OP, gives us: $AR \cdot OP \cdot PN = S \cdot OP^2$. Therefore, we conclude that:

$$AR \cdot BP \cdot CP, \text{ that is, } AR \cdot BP \cdot CQ = S \cdot OP^2.$$

Q.E.D.

Theorem

9 The area of any triangle ABC can be found, if the sides are subtracted separately from the semiperimeter (which is S) and the product of these three remaining sides is multiplied by the semiperimeter itself and the square root of the product is taken. That is, the area of triangle ABC

$$= \sqrt{S\,(S - AB)\,(S - AC)\,(S - BC)}\,.$$

Demonstration

Because of paragraph six, [we know that] the area of triangle ABC is equal to the product of the semiperimeter (S) and the radius OP of the inscribed circle. And so the area of triangle $ABC = S \cdot OP$. But, since we know from the preceding paragraph that $S \cdot OP^2 = AR \cdot BP \cdot CQ$, by multiplying S on both sides we get,

$$S^2 \cdot OP^2 = S \cdot AR \cdot BP \cdot CQ,$$

and here by taking the square root we get,

$$S \cdot OP = \sqrt{S \cdot AR \cdot BP \cdot CQ}$$

and therefore the area of triangle ABC equals

$$\sqrt{S \cdot AR \cdot BP \cdot CQ.}$$

But from paragraph seven it is evident that

$$AS = S - BC,\ \ BP = S - AC, and\ CQ = S - AB;$$

After substituting the values, we get:

$$\text{Area triangle } ABC = \sqrt{S\left(S - AB\right)\left(S - AC\right)\left(S - BC\right).}$$

Q.E.D.

Corollary 1

10. Here also an appropriate expression for the radius OP of a circle inscribed in a triangle can be shown. For since $S \cdot OP^2 = AR \cdot BP \cdot CQ$, we will have,

$$OP^2 = \frac{AR \cdot BP \cdot CQ}{S}, \text{ and therefore, } OP = \sqrt{\frac{AR \cdot BP \cdot CQ}{S}}.$$

Now after substituting AR, BP, and CQ for the values identified above, we get:

$$\text{The radius } OP \text{ of the inscribed circle} = \sqrt{\frac{\left(S - AB\right)\left(S - AC\right)\left(S - BC\right)}{S}}.$$

Corollary 2

11. Since S denotes the semiperimeter of the triangle, in such a way

$$S = \frac{1}{2}AB + \frac{1}{2}AC + \frac{1}{2}BC = \frac{1}{2}(AB + AC + BC),$$

by substituting the value, we get:

$$\begin{aligned} S - AB &= \frac{1}{2}AC + \frac{1}{2}BC - \frac{1}{2}AB = \frac{1}{2}\left(AC + BC - AB\right), \\ S - AC &= \frac{1}{2}AB + \frac{1}{2}BC - \frac{1}{2}AC = \frac{1}{2}\left(AB + BC - AC\right) \\ S - BC &= \frac{1}{2}AB + \frac{1}{2}AC - \frac{1}{2}BC = \frac{1}{2}(AB + AC - BC). \end{aligned}$$

Thus it will be:

$$S(S-AB)(S-AC)(S-BC)$$
$$= \frac{1}{16}(AB+AC+BC)(AC+BC-AB)(AB+BC-AC)(AB+AC-BC).$$

And, therefore, the area of a triangle can also be expressed in the following way:

$$\frac{1}{4}\sqrt{(AB+AC+BC)(AC+BC-AB)(AB+BC-AC)(AB+AC-BC)}$$

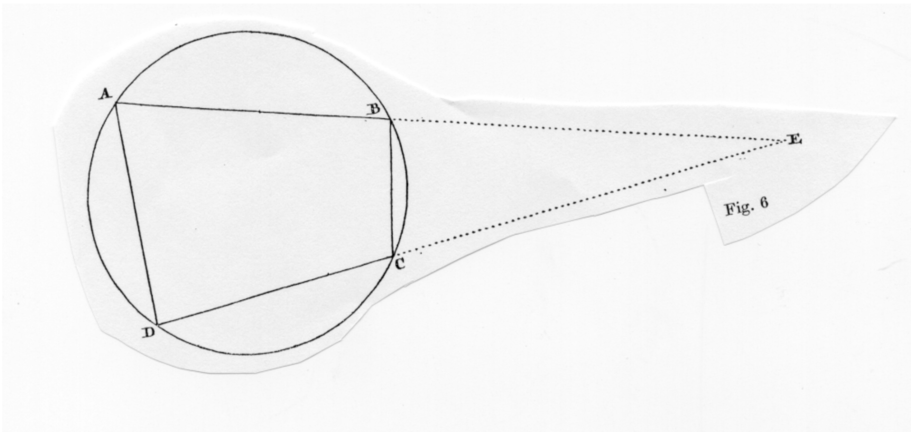
Scholion

12. This last formula for finding the area of any triangle is most well known and is commonly provided in the elements of geometry, although its demonstration can be attained with difficulty through the elements. There is also a rule for finding the area of any quadrilateral inscribed in a circle[4], which, in the same manner [as in Heron's formula], can be expressed rather conveniently by means of the sides alone. Indeed, if analysis is called in for support, its demonstration is not difficult, but for those who have attempted to do it in the manner common among geometers, the difficulties are great. Formerly, the illustrious Naudè did a good bit of work in this area and, in the *Berlin Miscellanies*[5], also provided a twin demonstration of this rule. However, Naudè's demonstration is not only exceedingly intricate with a multitude of lines covering the figure, so that indeed it cannot be understood without the greatest attention, but also the clear vestiges of analytical calculus everywhere create too much of a problem. Indeed, [fixing this problem] is my job in the propositions that are to follow.

Theorem

13. If two opposite sides AB and DC of quadrilateral $ABCD$ inscribed in a circle (Fig. 6) are extended until they meet at point E, the area of quadrilateral $ABCD$ will be to the area of triangle BCE as $AD^2 - BC^2$ is to BC^2.

[4] (Translator's Note) Now known as Brahmagupta's Rule.
[5] (Editor's Note) Phillip Naude, Jr., (1684-1747), in *Miscellaneis Berlinensibus*, vol. 5 (1737), p.10 and vol. 7 (1743), p.243.



Demonstration

Since both the angle BAD and the angle BCE are supplementary to angle BCD, it is the case that $BAD = BCE$ and similarly that $ADC = CBE$, whence the triangles AED and CEB are similar and therefore their areas will be related to one another just like the squares of corresponding sides, as, for instance, AD and BC. So, we have:

$$\triangle\ AED : \triangle\ CEB = AD^2 : BC^2$$

And by dividing, we get:

$$\triangle\ AED - \triangle\ CEB : \triangle\ CEB = AD^2 - BC^2 : BC^2$$

That is:

$$\square\ ABCD : \triangle\ CEB = AD^2 - BC^2 : BC^2.$$

Q.E.D.

Corollary 1

14. Therefore from the known area of triangle CEB, the area of quadrilateral $ABCD$ can be found, for we will have:

$$\square\ ABCD = \frac{AD^2 - BC^2}{BC^2} \cdot \triangle\ BEC$$

Or, if for the sake of brevity the area of triangle BEC is designated by the letter T and the area of quadrilateral $ABCD$ is designated by the letter Q, we will have:

$$Q = \frac{AD^2 - BC^2}{BC^2} \cdot T.$$

Corollary 2

15. Then, since $AD^2 - BC^2 = (AD + BC)(AD - BC)$ is the difference of the squares, we get

$$\frac{AD^2 - BC^2}{BC^2} = \frac{AD - BC}{BC} \cdot \frac{AD + BC}{BC}.$$

And hence will we have this equation:

$$Q = \frac{AD - BC}{BC} \cdot \frac{AD + BC}{BC}\ T,$$

After we take the squares, this equation becomes:

$$QQ = \frac{AD - BC}{BC} \cdot \frac{AD - BC}{BC} \cdot \frac{AD + BC}{BC} \cdot \frac{AD + BC}{BC} \cdot T \cdot T.$$

Corollary 3

16. Also from paragraph 11 above, we gather that the area T of triangle BEC is

$$\frac{1}{4}\sqrt{(BE + CE + BC)\,(BE + CE - BC)\,(BE - CE + BC)\,(CE - BE + BC)},$$

Whence TT is

$$\frac{1}{16}\,(BE + CE + BC)\,(BE + CE - BC)\,(BE - CE + BC)\,(CE = BE + BC)\,.$$

Hence, the value of the square of the area of quadrilateral $ABCD$—that is, of QQ itself—will appear by combining these factors of TT with the ones found before:

$$QQ = \frac{1}{16} \cdot \left[\begin{array}{c} \frac{(AD-BC)(BE+CE+BC)}{BC} \cdot \frac{(AD-BC)(BE+CE-BC)}{BC} \\ \frac{(AD+BC)(BC+BE-CE)}{BC} \cdot \frac{A(D+BC)(BC-BE+CE)}{BC} \end{array} \right].$$

Corollary 4

17. This form can be stated in such a way that we may say that the square of the area of $ABCD$ multiplied by 16 (that is, $16QQ$) is equal to the product of these four factors:

$$\text{I.....} \frac{(AD - BC)(BE + CE + BC)}{BC}$$
$$\text{II.....} \frac{(AD - BC)(BE + CE - BC)}{BC}$$
$$\text{III.....} \frac{(AD + BC)(BC + BE - CE)}{BC}$$
$$\text{IV.....} \frac{(AD + BC)(BC - BE + CE)}{BC}$$

Theorem

18. With these things in place, which were assumed in the preceding theorem, we have:

$$BE + CE : BC = AB + CD : AD - BC$$

Demonstration

Since triangles BEC and DEA are similar, we have:

$$BE : DE = BC : AD \text{ and likewise } CE : AE = BC : AD;$$

Whence, by dividing on both sides, we get:

$$\begin{aligned} BE &: DE - BE = BC : AD - BC, \\ CE &: AE - CE = BC : AD - BC. \end{aligned}$$

Since, therefore, the ratio of BE to $DE - BE$ is the same as the ratio of CE to $AE - CE$—as to be sure is BC to $AD - BD$—the ratio of the sum of the antecedents $BE + CE$ to the sum of the consequents, together with $AE - CE$, will also be the same. This will be:

$$BE + CE : DE - BE + AE - CE = BC : AD - BC$$

But

$$DE - BE + AE - CE = DE - CE + AE - BE = BE + AB$$

And thus we will get:

$$BE + CE : AB + CD = BC : AD - BC$$

And by alternating, it becomes:

$$BE + CE : BC = AB + CD : AD - BC.$$

Q.E.D.

Corollary 1

19. Therefore, since

$$BE + CE : BC = AB + CD : AD - BC$$

by "putting together"[6] we get:

$$BE + CE + BC : BC = AB + CD + AD - BC : AD - BC$$

Whence, the product of the outer parts will be equal to the product of the middle parts; that is,

$$(AD - BC)\ (BE + CE + BC) = BC\ (AB + CD + AD - BC)$$

And hence the first of the factors presented in paragraph 17 will be:

$$\text{I.} \quad \frac{(AD - BC)\,(BE + CE + BC)}{BC} = AB + CD + AD - BC.$$

Corollary 2

20. In a similar way, by proportion:

$$BE + CE : BC = AB + CD : AD - BC$$

By dividing we get:

$$BE + CE - BC : BC = AB + CD - AD + BC : AD - BC.$$

Whence the following rectangles will be similar:

$$(AD - BC)\,(BE + CE - BC) = BC\,(AB + CD - AD + BC)\,,$$

And, therefore, the second of the factors presented in paragraph 17 will be:

$$\text{II.} \quad \frac{(AD - BC)\,(BE + CE - BC)}{BC} = AB + CD - AD + BC.$$

[6](Translator's Note) *Componendo* is Euler's term for making use of the following fact, and other similar facts, concerning proportions: If

$$\frac{x}{y} = \frac{z}{w}$$

then

$$\frac{x}{y + x} = \frac{z}{w + z}.$$

Theorem

21. With these ideas in place, it is clear that if the two sides AB and DC of a quadrilateral $ABCD$ inscribed in a circle are extended out until they meet at point E, we will get:

$$CE - BE : AB - DC = BC : AD + BC.$$

Demonstration

The similar triangles BCE and DEA[7], as noted before, display these proportions:

$$BE : DE = BC : AD \text{ and } CE : AE = BC : AD,$$

From these one gets, respectively, by "putting together:

$$\begin{aligned} BE &: DE + BE + BC : AD + BC \\ CE &: AE + CE = BC : AD + BC. \end{aligned}$$

Therefore, since the ratio of BE to $DE + BE$ is the same as the ratio of CE to $AE + CE$, the ratio of the difference of the antecedents $CE - BE$ to the difference of the consequents $AE + CE$, less $DE + BE$, is also the same as the ratio of BC to $AD + BC$; that is,

$$CE - BE : AE + CE - DE - BE = BC : AD + BD.$$

But

$$AE + CE - DE - BE = AE - BE - DE + CE = AB - CD$$

And thus we get:

$$CE - BE : AB - CD = BC : AD + BC$$

And by alternating:

$$CE - BE : BC = AB - CD : AD + BC.$$

Q.E.D.

Corollary 1

22. Thus, since by reversing we get,

$$BC : CE - BE = AD + BC : AB - CD,$$

[7] (Translator's Note) It would be better to denote this triangle as DAE, to preserve correspondence of vertices.

by "putting together" we will get:

$$BC + CE - BE : BC = AD + BC + AB - CD : AD + BC.$$

And, with the area of the middle and extreme parts equal, it becomes:

$$(AD + BC)(BC + CE - BE) = BC(AD + BC + AB - CD).$$

Thus the fourth of the factors presented in paragraph 17 will be:

$$\text{IV.} \quad \frac{(AD + BC)\,(BC - BE + CE)}{BC} = AB + AD + BC - CD.$$

Corollary 2

23. The same can be done by proportion:

$$BC : CE - BE = AD + BC : AB - CD$$

By dividing we get:

$$BC - CE + BE : BC = AD + BC - AB + CD : AD + BC.$$

And hence, we will get:

$$(AD + BC)(BC + BE - CE) = BC(AD + BC + CD - AB).$$

Whence the third of the factors presented in paragraph 17 will be:

$$\text{III.} \quad \frac{(AD + BC)\,(BC + BE - CE)}{BC} = AD + BC + CD - AB$$

Theorem

24. The area of quadrilateral $ABCD$ inscribed in a circle can be found by subtracting each of the sides one at a time from the semiperimeter, then multiplying the four remaining differences together and taking the square root of the product.

Demonstration

Extend two opposite sides (AB, CD) until they meet at some point E and designate the area of quadrilateral $ABCD$ as Q. We saw in paragraph 17 that the value $16QQ$ was equal to the product of four factors. We expressed these factors more succinctly in paragraphs 19 and 20 and in paragraphs 22 and 23, such that now the value of $16QQ$ itself is equal to the product of these four factors:

$$\begin{aligned}
\text{I.} \quad \frac{(AD-BC)\,(BE+CE+BC)}{BC} &= AB+CD+AD-BC, \\
\text{II.} \quad \frac{(AD-BC)\,(BE+CE-BC)}{BC} &= AB+CD-AD+BC, \\
\text{III.} \quad \frac{(AD+BC)\,(BC+BE-CE)}{BC} &= AD+BC+CD-AB, \\
\text{IV.} \quad \frac{(AD+BC)\,(BC-BE+CE)}{BC} &= AB+AD+BC-CD.
\end{aligned}$$

Here, therefore, $16QQ$ will be equal to this product

$$\begin{gathered}
(AB+CD+AD-BC)(AB+CD+BC-AD) \\
(AD+BC+CD-AB)(AB+AD+BC-CD).
\end{gathered}$$

But if we set the sum of all the sides $AB+BC+BC+DA$ equal to $2S$, such that half the sum is S, we will get:

$$\begin{aligned}
2S-2AB &= BC+CD+DA-AB = \text{ factor III}, \\
2S-2BC &= AB+CD+DA-BC = \text{ factor I}, \\
2S-2CD &= AB+BC+DA-CD = \text{ factor IV}, \\
2S-2DA &= AB+BC+CD-DA = \text{ factor II}.
\end{aligned}$$

Whence the product of these four factors will be:

$$(2S-2AB)(2S-2BC)(2S-2CD)(2S-2DA).$$

After factoring out each 2, we get:

$$16\,(S-AB)\,(S-BC)(S-CD)(S-DA),$$

to which, therefore, the value of $16QQ$ itself is equal. So, by dividing by 16 on both sides, we get:

$$QQ=(S-AB)(S-BC)(S-CD)(S-DA).$$

Therefore, after taking the square root, we get:

$$Q = \text{ Area } ABCD = \sqrt{(S-AB)(S-BC)(S-CD)(S-DA)}.$$

It is clear, then, that the area of quadrilateral $ABCD$ can be found if we subtract the sides one at a time from the semiperimeter S, multiply together the four remaining differences $S-AB, S-BC, S-CD$, and $S-DA$, and take the square root of the product. Q.E.D.

Scholion

25. Having demonstrated these theorems concerning the area of a triangle and a quadrilateral inscribed in a circle—which themselves are indeed quite well known—I add another theorem which, up to now, has neither been proposed nor demonstrated. It involves a remarkable property of general quadrilaterals[8] one that is most noteworthy. Just as in all parallelograms the sum of the squares of both diagonals is equal to the sum of the squares of the four sides, so I will demonstrate that in all quadrilaterals that are not parallelograms the sum of the squares of both diagonals is always less than the sum of the squares of the four sides and, moreover, that the defect (the difference between the square of the diagonals and the sum of the squares of the sides) can be easily assigned.

Theorem

26. For any quadrilateral $ABCD$ (Fig. 7) with diagonals AC and BD, if along both sides AB and BC we construct a parallelogram $ABCE$ that shares three points A, B, and C with the quadrilateral and then connect points D and E with line DE, the sum of the squares of the sides of the quadrilateral $AB^2 + BC^2 + CD^2 + DA^2$ will be greater than the sum of the squares of the diagonals $AC^2 + BD^2$. Moreover, the excess will be equal to the square of the line DE. That is,

$$AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2 + DE^2.$$

[8](Translator's Note) That is, any quadrilateral, not necessarily cyclic.

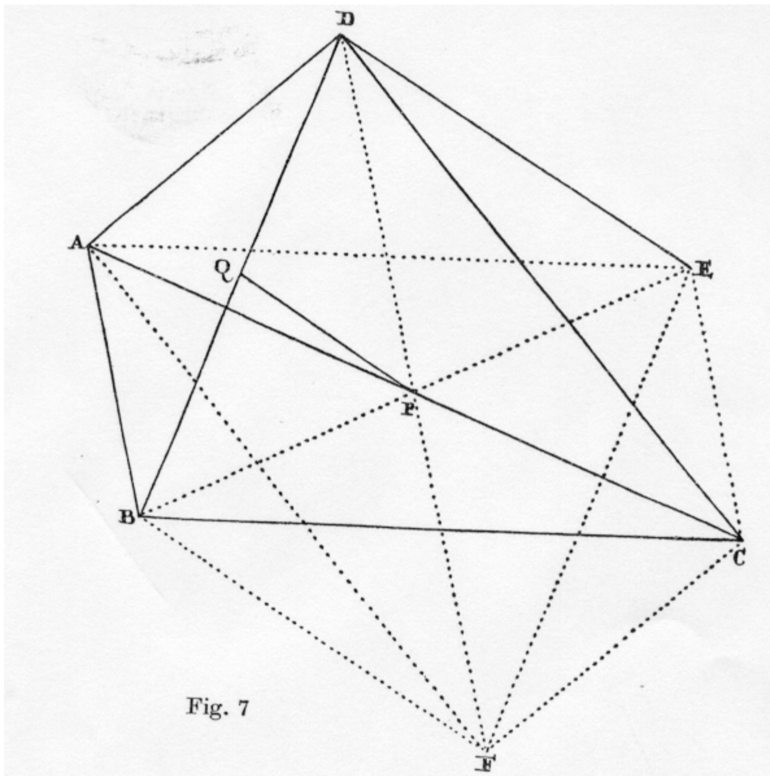


Fig. 7

Demonstration

In parallelogram $ABCE$, draw the other diagonal BE: the one that is not shared with the quadrilateral. Then, draw CF parallel and equal to AD and BF parallel and equal to ED. Now because $BC = AE$, these lines will meet at point F[9], such that triangle CBF will be similar and equal to triangle AED. After this, connect lines AF, DF, and EF. From here it is manifest that $ADCF$ and $BDEF$ are parallelograms and that the diagonals of the former are AC and DF while the diagonals of the latter are BE and DF. Therefore, by a known property of parallelograms, we get:

$$\begin{aligned} \text{from } ADCF\ldots\ 2AD^2 + 2CD^2 &= AC^2 + DF^2 \\ \text{from } BDEF\ldots\ 2BD^2 + 2DE^2 &= BE^2 + DF^2 \end{aligned}$$

[9] (Translator's Note) Euler appears to mean that, if one draws the line through C parallel to AD, and the line through B parallel to DE, then these lines must meet at a point F so that triangles CBF and AED are similar.

Whence by solving each side of the equation for DF^2, we have:

$$2AD^2 + 2CD^2 - AC^2 = 2BD^2 + 2DE^2 - BE^2 = DF^2.$$

And by adding AC^2 on both sides it becomes:

$$2AD^2 + 2CD^2 - AC^2 = 2BD^2 + 2DE^2 - AC^2 - BE^2.$$

Now, from the nature of parallelogram $ABCE$, we have:

$$2AB^2 + 2BC^2 = AC^2 + BE^2.$$

This equation, when added to the former, gives us:

$$2AD^2 + 2CD^2 + 2AB^2 + 2BC^2 = 2BD^2 + 2DE^2 + 2AC^2.$$

And after dividing by 2, we get:

$$AD^2 + CD^2 + AB^2 + BC^2 - BD^2 + DE^2 + AC^2$$

That is,

$$AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2 + DE^2.$$

But AB, BC, CD, and DA are the four sides of the proposed quadrilateral $ABCD$, and AC and BD are its diagonals. Whence the sum of the squares of the sides is equal to the sum of the squares of both diagonals plus the square of the segment DE. [The existence of] this segment shows how a quadrilateral differs from a parallelogram. Q.E.D.

Corollary 1

27. Therefore, the more a quadrilateral differs from a parallelogram—that is, the greater the interval DE—the more the sum of the squares of the sides of the quadrilateral will exceed the sum of the squares of the diagonals.

Corollary 2

28. Therefore, since in every parallelogram the sum of the squares of the sides is equal to the sum of the squares of the diagonals, in every true quadrilateral which is not a quadrilateral [the sum] is greater [than the sum of the squares of the diagonals] it follows that there can be no quadrilateral in which the sum of the squares of the sides is less than the sum of the squares of the diagonals.

Corollary 3

29. If each diagonal AC and BD of the proposed quadrilateral $ABCD$ is bisected, the former at point P and the latter at point Q, then the straight line PQ will equal half the interval DE, and DE^2 will be equal to four times

the square of the line PQ. Consequently, the difference between the sum of the squares of the sides and the sum of the squares of the diagonals will be equal to four times square of the line PQ.

Corollary 4

Therefore, the theorem proposed can be stated in the following way without mention of another parallelogram: In ever quadrilateral $ABCD$ (Fig. 8), if its diagonals AC and BD are bisected at points P and Q and these points are connected by the line PQ, the sum of the squares of the sides $AB^2 + BC^2 + CD^2 + DA^2$ will be equal to the sum of the squares of the diagonals $AC^2 + BD^2$ plus four times the square of the line PQ; that is, $AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2 + 4PQ^2$.

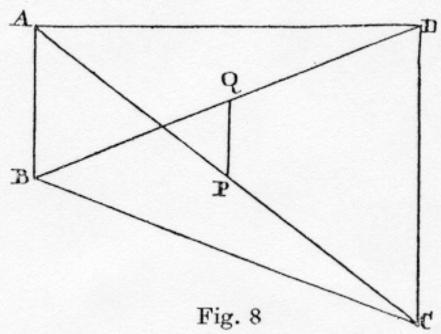


Fig. 8