Translation of Euler's paper

E105 -- Memoire sur la plus grande equation des planetes

(Memoir on the Maximum value of an Equation of the Planets)

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Introduction to the translation:

This translation is the result of a fortunate collaboration between student and professor. Jasen Scaramazza is an undergraduate mathematics and physics major at Rowan University. Tom Osler has been a mathematics professor for 49 years. Together we struggled to understand this astronomical work.

Jasen translated Euler's paper from French to English and both he and Tom worked to comprehend Euler's achievement. When translating Euler's words, we tried to imagine how he would have written had he been fluent in modern English and familiar with today's mathematical jargon. Often he used very long sentences, and we sometimes converted these to several shorter ones. However, in almost all cases we kept his original notation, even though some is very dated. We thought this added to the charm of the paper.

Euler was very careful in proof reading his work, and we found few typos. When we found an error, we called attention to it (*in parenthesis using italics*) in the body of the translation. Other errors are probably ours. We also made a few comments of our own within the translation identifying our words in the same way.

The separate collection called Notes was made while translating this paper. This is material that we accumulated while trying to understand and appreciate Euler's ideas. In these notes we completed some steps that Euler omitted, introduced some modern notation and elaborated on a few of Euler's derivations that we found too brief. We also added a few simple ideas and comments of our own not found in the text. A separate Synopsis is also available and was made from a condensation of the Notes.

I. In astronomy, we learn about the planets' equations of the center from a heliocentric standpoint, in which an observer is viewing them from the sun. This is needed since an irregularity is observed as we watch planets move across the sky from the Earth. Sometimes they move faster, then slower, and even stop and appear motionless at the same points in space. They will sometimes even turn around and begin retrograde motion. However, astronomers have realized that if planetary movements were observed from the sun, these irregularities disappear almost entirely. An observer placed in the star would not even see the planets stop or enter retrograde motion. They will see an unchanging path directly following the order of the constellations. Nevertheless, this movement would not be quite uniform. There would still inherently be a variation in speed. The same planet would sometimes be observed moving faster and sometimes slower. It is this changing motion that astronomical tables have designated "The Equation of the Center ."

II. Planets as seen from the Earth do not seem to follow any laws. We consider our distance from them to be the cause, although it is very difficult to determine distance solely by observation. But if the movements of the planets with respect to the sun are recorded, and are represented from a supposed observer in the sun, then anomalies in planetary distances will almost be eliminated. This is because in each revolution, each planet will only once be at its greatest distance from the sun, and once at its closest proximity; moreover, these two points will be diametrically opposed to each other and unchanging in the sky. The time intervals during which the planet travels from the farthest distance, to the closest and back will be constant.

The point in the sky in which the planet appears the farthest from the sun is called its *Aphelion* and the opposite point where it is closest is called its *Perihelion*. Additionally, the time in which the planet leaves from the Aphelion or from the Perihelion, and returns back is called its period.

III. The changing distances of each planet from the sun nicely follow a constant relation with the varying movement, as seen from the sun. When the planet is farther away, it moves slower, and when it approaches the sun, it speeds up. This is the beautiful law that Kepler discovered, and that Newton has since demonstrated by the principles of mechanics. More specifically, each planet in equal times sweeps not equal angles around the sun, but equal areas. The law also provides irrefutable evidence that planetary orbits are ellipses with the sun at one of the foci. The variation in movement is clearly regulated. Equal areas in equal times are swept in each ellipse, drawing the areas by lines straight from the planet to the sun.

IV. The first thing that we can infer from this rule is that the greater the distance between the farthest and closest points of a planet to the sun, the more irregular its movement would appear from the sun. Whereas if a planet would always maintain the same distance from the sun, this is to say that its orbit would be a circle with the sun as the center, then its movement would be so regular that in equal times, it would sweep out not only equal areas, but also equal angles. In this case, we would therefore be able to very easily determine, by the third law (*Kepler's Third Law*) the distance from the planet to the sun for any time. But as this condition does not exist for any planet, we have the custom to ideally conceive for each planet another planet that serves as a companion. It completes its revolution around the sun in the same time period, but with uniform movement. We furthermore suppose that this fictitious planet would appear at the same point in the sky as the true when it is at the Aphelion and the Perihelion. After these two planets have passed by the aphelion, the false planet will appear to go faster than the true and the real planet will imperceptibly increase its speed until it will have caught the false one at the Perihelion. Then it will pass its partner in speed, and will leave it behind until they rejoin again at the Aphelion. Therefore, with the exception of these two points, the two planets will be perpetually separated from each other, and the difference between the positions of the two is what we call the equation of the center of the planet, or the Prostapherese. As it is easy to find the position of the false planet at any time, if the equation that astronomical tables provide is known, we will know the position of the true planet.

V. Astronomers call the *mean anomaly* either the distance from the false planet to the aphelion, or the angle that the false planet forms with the aphelion and the sun. We can easily determine it by the time that elapsed since the planet's passage from the aphelion. The true anomaly is the distance from the true planet to the aphelion, or the angle that the true planet forms with the aphelion and the sun. Therefore, when the planet moves from the aphelion to the perihelion, we find the true anomaly in subtracting the value from the equation of the center from the mean anomaly; conversely, when the planet returns from the perihelion to the aphelion, we must add the value from equation to the mean anomaly in order to have the true anomaly. Thus, we can determine by the mean anomaly or by the true, the actual distance from the planet to the sun. Consequently, if we determine the position of the Earth seen from the sun for the same time, trigonometry shows where the planet as seen by the Earth must appear, also known as its geocentric location.

VI. When the planet is at the perihelion or the aphelion, the (*value of the*) equation of the center is zero. It is necessary to realize that the (*value of the*) equation grows as the planet moves, and later it decreases again. There will therefore be a place where the equation will have a maximum. Several very important astronomical questions arise here: What is the maximum for each planet? And what mean anomaly corresponds to this maximum? Moreover, as the greatest value of the equation is determined by the eccentricity of the planet's orbit, which is the fraction whose numerator is the distance between the foci of the ellipse, and whose denominator is the major axis of the ellipse? And conversely, we will determine the eccentricity from the maximum value of the equation. I am therefore going to examine these questions, for which a rigorous mathematical solution still does not exist.

VII. Let half the length of the major axis of the orbit of each planet = a, which is customarily called in astronomy the average distance from the planet to the sun. The eccentricity, or the distance between the foci divided by the major axis, = n, and shrinks if the orbit becomes a circle. It becomes larger when the orbit elongates from a circle. And if the elongation goes to infinity, meaning that the orbit becomes a parabola, then the eccentricity n will become equal to one. But if it becomes a hyperbola, it will be larger than one. The major axis being $2a$, the distance between the foci will be = 2*an*, and the distance from each focus to the center = *an*. Consequently, the distance from the aphelion to the sun will be = $a + an = a(1+n)$ and the distance from the perihelion to the sun = $a - an = a(1-n)$. Then the semi-minor axis will be = $a\sqrt{1-n^2}$ and the parameter's half will be $a(1-n^2)$. (*We do not understand this highlighted phrase.*)

VIII. These things being supposed, for a given time let the passage of the planet past the aphelion, have the mean anomaly = *x*, and the corresponding true anomaly = *z*. The (*value of*) equation of the center, as we have seen, will be = $x - z$. Let the distance from the planet to the sun = *r*. To express the relation between the mean anomaly and the true anomaly, it will be necessary to employ a new angle holding a kind of middle between *x* and *z*, which Kepler named the eccentric anomaly. Let this eccentric anomaly = *y*. Elsewhere, I determined, with the mean anomaly *x*, the true anomaly *z*, and with the distance denoted by *r*, that without question we have *x* = $y + n\sin y$; $\cos z = \dfrac{n+\cos y}{1+n\cos y}$; and $r = a(1+n\cos y)$. Consequently, we will have $\sin z = \dfrac{\sin y\sqrt{1-n^2}}{1+n\cos y}$; and $\tan z = \dfrac{\sin y\sqrt{1-n^2}}{n+\cos y}$; from which, by the eccentric anomaly *y*, we can find the mean anomaly and the true, and the distance from the planet to the sun. With these formulas, it is easy to calculate the Astronomical Tables for planetary movements.

IX. Before finding the maximum value of the equation, it will be convenient to start with the following problem that can have some use in astronomy:

Find the mean anomaly and the true, when the distance from the planet to the sun is equal to the average distance a.

As here *r* must be = *a*, it will be necessary that $\cos y = 0$, and it follows that the eccentric anomaly *y* will be = 90°. We then compute from the average anomaly (*Kepler's equation*) *x* = 90° + *n*. For this computation, the addition of the sine term, with the radius here equal to one, we must look in a similar circle the arc = *n* (*same number as angle in radians*), and the angle (*converted to degrees*) that measures this arc must be added to 90° to find the sought average anomaly *x*. (*See Notes for more details*.) Or, as *n* is a number less than one, we must treat it as a sine, and subtract 4.6855749 from its logarithm. The corresponding number to the logarithm that remains will provide the angle *n* expressed in seconds. But the true anomaly *z* that corresponds to this eccentric anomaly *y* = 90° will be such that $\cos z$ = *n*, and *z*= $A\cos n$ = 90° - $A\sin n$. (*Euler uses* Acos *to mean the inverse function* arccos.) Let *m* be the angle for which the sine = *n*, and we will have *z* = 90°- *m* and the center equation in this case will be = *n* + *m* = *n* + $A\sin n$. This is why if the distance from the planet to the sun *r* is equal to the average distance to the sun, what happens when the planet is in the conjugated axis (*minor axis vertex*) of its orbit, the mean anomaly *x* will be = 90°+ A$\sin n$, and the center equation = *n* + $A\sin n$.

X. I have begun with this problem because in this case, the expression $n + A\sin n$ hardly differs significantly from the maximum of the equation (*of the center*), when the eccentricity *n* is very small, which nearly happens in all planets. I presented in the Dissertation on Planetary Movements & The Orbit of the Sun, the

T.VII. of the Memoires of the Academy of Petersburg, the following solution of the problems on the maximum of the equation. The true anomaly, as I showed in this paper, can be expressed by an infinite series, in the following manner:

$$z = y - n\sin y + \frac{1}{4}n^2\sin 2y - \frac{1}{3.4}n^3(\sin 3y + 3\sin y) + \frac{1}{4.8}n^4(\sin 4y + 4\sin 2y) - \frac{1}{5.16}n^5(\sin 5y + 5\sin 3y + 10\sin y) + \frac{1}{6.32}n^6(\sin 6y + 6\sin 4y + 15\sin 2y) \ldots \& \ etc.$$

If n is a very small fraction, z will be approximately = $y - n\sin y$ and because x is = $+n\sin y$, (*should be* $x = y + n\sin y$) the value of the equation will be = $2n\sin y$. The equation will be greatest when y is = 90°, in which case r becomes = a. However if the eccentricity of the planet is not so small, as happens with Mercury, this evaluation will deviate a little from the truth. This is particularly important when determining the period of some comet and recording the comet's movement in the tables as we do the planets. This calculation will deviate very much from the truth, for the maximum of the equation will occur considerably far from the place where the distance of the planet to the sun is equal to half the major axis.

XI. For all these cases, we must deduce the maximum of the equation by using the method of maximums and minimums, rather than using the formulas that are only approximately true. Thus, to determine the average anomaly and the true, we must first know the eccentric anomaly. I will start with the following problem.

Being given the eccentricity of a Planet n, find the eccentric anomaly that corresponds to the maximum of the equation of the center.

Let the mean anomaly = *x*, and the true anomaly = *z*, then the center equation is = $x - z$. This equation will become the greatest when $dx - dz = 0$, or when $dx = dz$. Calling the eccentric anomaly *y*, we will have as we have seen earlier, $x = y + n\sin y$ and $\cos z = \frac{n + \cos y}{1 + n\cos y}$. Differentiating results in $dx = dy + ndy\cos y$, and $-dz\sin z = \frac{-dy\sin y + n^2 dy\sin y}{(1 + n\cos y)^2}$, or just as well $dz\sin z = \frac{(1 - n^2)dy\sin y}{(1 + n\cos y)^2}$. (*See section VIII.*) But $\sin z$ is = $\frac{\sqrt{1 - n^2}\sin y}{1 + n\cos y}$, and therefore we have $dz = \frac{dy\sqrt{1 - n^2}}{1 + n\cos y}$. Since *dx* must be equal to *dz*, we obtain the equation: $1 + n\cos y = \frac{\sqrt{1 - n^2}}{1 + n\cos y}$, and then we get $1 + n\cos y = \sqrt[4]{1 - n^2}$ and $\cos y = \frac{\sqrt[4]{1 - n^2} - 1}{n}$. Let $y = 90° + \lambda$, and we will have $\sin\lambda = \frac{1 - \sqrt[4]{1 - n^2}}{n}$, or $\sin\lambda = \frac{n}{(1 + \sqrt[4]{1 - n^2})(1 + \sqrt{1 - n^2})}$, from which it seems that the eccentric anomaly is a little greater than in the previous case, where it was y = 90°.

XII. As before we let $n = \sin m$, and we now have $\sqrt{1 - n^2} = \cos m$. Thus given the eccentricity, the angle *m* will be known. We will therefore have $\sin\lambda = \frac{1 - \sqrt{\cos m}}{\sin m}$, and $\cos\lambda = \frac{\sqrt{2\sqrt{\cos m} - \cos m - \cos m^2}}{\sin m}$. But if the eccentricity *n* is much less than one, as happens in all planets, we will have

$\sqrt[4]{1-n^2} = 1 - \frac{1}{4}n^2 - \frac{1*3}{4*8}n^4 - \frac{1*3*7}{4*8*12}n^6 - \frac{1*3*7*11}{4*8*12*16}n^8$ etc. It follows that the angle λ, by which the eccentric anomaly y surpasses being a right angle, will be expressed as $\sin\lambda = \frac{1}{4}n + \frac{1*3}{4*8}n^3 + \frac{1*3*7}{4*8*12}n^5 + \frac{1*3*7*11}{4*8*12*16}n^7 + \text{etc.}$ Thus by knowing the eccentricity, we can easily find the angle λ, and then the eccentric anomaly $y = 90° + \lambda$. The cosine of the angle λ is therefore

$$\cos\lambda = 1 - \frac{1}{32}n^2 - \frac{49}{2048}n^4 - \frac{1233}{65536}n^6 - \text{etc.}$$

XIII. Having solved the problem of determining the eccentric anomaly y corresponding to the maximum value of our equation, we can determine the corresponding mean and true anomalies. But it is expedient to find each of them separately.

Being given the eccentricity n, find the mean anomaly that corresponds to the maximum value of the equation.

The eccentric anomaly for this case is $y = 90° + \lambda$ and $\sin\lambda = \frac{1-\sqrt[4]{1-n^2}}{n}$. Because $x = y + n\sin y$, we will have $x = 90° + \lambda + n\cos\lambda$. But if we want to express the excess of this angle above 90° by n, since λ is equal to $\sin\lambda + \frac{1}{6}\sin\lambda^3 + \frac{3}{40}\sin\lambda^5 + etc.$, we will have $\lambda = \frac{1}{4}n + \frac{37}{384}n^3 + \frac{2363}{40960}n^5 + etc.$, whose value being substituted in place of λ and of $\cos\lambda$ found above, the mean anomaly will be

$x = 90^\circ + \frac{5}{4}n + \frac{25}{384}n^3 + \frac{1383}{40960}n^5 + etc.$. But if n is not a quantity so small that these series converge rapidly enough, then it will suffice to use the expression previously found $x = 90^\circ + \lambda + n\cos\lambda$, which is easy to work with in calculus.

XIV. Before the maximum of the equation of the center can be determined, we must also find the true anomaly.

Being given the eccentricity, find the true anomaly that corresponds to the maximum of the equation.

The eccentric anomaly in this case is found to be $y = 90^\circ + \lambda$ knowing that $\sin\lambda = \frac{1-\sqrt[4]{1-n^2}}{n}$. Consequently, when we let the true anomaly equal z, we will have $\cos z = \frac{n+\cos y}{1+n\cos y} = \frac{n+\cos y}{\sqrt[4]{1-n^2}} = \frac{n-\sin y}{\sqrt[4]{1-n^2}}$. The cosine being positive shows that $z < 90°$. Let $z = 90^\circ - \mu$, and it will be that

$\sin\mu = \frac{n-\sin\lambda}{\sqrt[4]{1-n^2}} = \frac{n^2-1+\sqrt[4]{1-n^2}}{n\sqrt[4]{1-n^2}} = \frac{1}{n} - \frac{1}{n}\sqrt[4]{(1-n^2)^3}$. Thus, by knowing the eccentricity, we will find the angle μ. But if n is a very small fraction, we will almost have $\sqrt[4]{(1-n^2)^3} = 1 - \frac{3}{4}n^2 - \frac{3*1}{4*8}n^4 - \frac{3*1*5}{4*8*12}n^6 - \frac{3*1*5*9}{4*8*12*16}n^8 - etc.$, from which we will get $\sin\mu = \frac{3}{4}n + \frac{3}{32}n^3 + \frac{5}{128}n^5 + \frac{45}{2048}n^7 + etc.$. And the same angle μ will determine itself by the formula: $\mu = \sin\mu + \frac{1}{6}\sin\mu^3 + \frac{3}{40}\sin\mu^5 + etc.$ We will then have $\mu = \frac{3}{4}n + \frac{21}{128}n^3 + \frac{3409}{40960}n^5 + etc.$

XV. Now if we subtract the true anomaly from the mean anomaly, we will have the maximum value of the equation of the center.

Being given the eccentricity of the planet's orbit, find the maximum of x - z.

For this maximum, we have found the mean anomaly

$$x = 90^\circ + \lambda + n\cos\lambda \text{ with } \sin\lambda = \frac{\sqrt[4]{1-n^2}}{n},$$

and we have also found the true anomaly

$$z = 90^\circ - \mu \text{ with } \sin\mu = \frac{1}{2}\sqrt[4]{(1-n^2)^3}.$$

The maximum will be = $\lambda + \mu + n\cos\lambda$. But if in the case where n is a small enough fraction, we want only a maximum that approaches the truest possible value. From the above results we have:

$2n + \frac{11}{48}n^3 + \frac{599}{5120}n^5 + etc.$ But when the distance from the planet to the sun is equal to half the major axis, the equation is = $n + A\sin n = 2n + \frac{1}{6}n^3 + \frac{3}{40}n^5 + etc.$ Thus the maximum surpasses this by a quantity = $\frac{1}{16}n^3 + \frac{43}{1024}n^5 + etc.$

XVI. Since we have found $1+n\cos y = \sqrt[4]{1-n^2}$, the distance from the planet to the sun when its equation is at a maximum, will be $r = a\sqrt[4]{1-n^2}$, a distance that is always less than half of the major axis. From this we can easily determine using the eccentricity, the mean anomaly and the distance from the planet to the sun that corresponds to the maximum. But if the maximum is given as *m*, and if we want to reciprocally find the eccentricity *n*, the problem becomes very difficult, and can only be found by approximation. Because we found the equation $m = \lambda + \mu + n\cos\lambda$, by which we must find the value of *n*; there is no other way to solve the equation than to try different values of *n*, and deducing the maximum from there. We will discover in fact by this method first the boundaries between which the true value of *n* is contained, and following the same route, we will make the limits still closer, until finally by the rules of interpolations we can calculate the true value of the eccentricity *n*.

XVII. But if the eccentricity is not very large, (meaning our approximate formulas can be used without error) we will be able to directly find the eccentricity by the given maximum.

Being given the maximum, find the eccentricity of the planet's orbit.

Let the maximum = *m*, and the eccentricity = *n*, we will have

$m = 2n + \frac{11}{48}n^3 + \frac{599}{5120}n^5 + etc.$, from which we calculate by inversion

$n = \frac{1}{2}m - \frac{11}{768}m^3 - \frac{587}{2^{16}*15}m^5 - etc.$ Here we must express the greatest value of the equation *m* in fractions of the radius (*this means in radians rather than degrees*), which is done by converting the angle *m* into seconds. To the logarithm of the number (*m in seconds*) we add 4.6855749, because we will thus have the logarithm of the number *m* (*in radians*). The mean anomaly *x* that corresponds to the maximum will be $x = 90^{\circ} + \frac{5}{8}m - \frac{5}{2^9 * 3}m^3 - \frac{1}{2^9 * 5}m^5 - etc.$

We will approach closely enough to the mean anomaly, if at 90 degrees we add five eighths of the maximum.

XVIII. To clarify the application of these solutions to astronomical calculation, we will take for an example the orbit of Mercury. The astronomical tables have the eccentricity equal to $\frac{797}{3871}$. We will then have $n = 0.20589; \log n = 9.3136351$. If the distance from Mercury to the sun is equal to its semi-major axis, or if we make the eccentric anomaly = 90°, the mean anomaly *x* will become = $90^{\circ} + n$. From this, we find the angle *n*. From $\log n = 9.3136351$, we subtract 4.6855749 to get this logarithm: 4.6280602, which corresponds to the number 42468”. From this we have *n* = 11°.47’. 48”. Thus the mean anomaly is $x = 3^{s}.11^{\circ}.47'.48''$. (*The symbol “*$3^{5}$ *” means 90 degrees.*)But the true anomaly in this case is $z = 90^{\circ} - A\sin n$. Now $A\sin n = 11^{\circ}.52'.54''$, from which we get $z = 90^{\circ} - 11^{\circ}.52'.54''$. From this the equation becomes = 23°.40’.42”, which is nearly two minutes less than the maximum.

XIX. But to find the maximum, we make the following calculations

$\log n^2 = 8.6272702$

$n^2 = 0.0423906$

$1 - n^2 = 0.9576093$

$\log(1 - n^2) = 9.981183$

$\log(\sqrt[4]{1 - n^2}) = 9.9952971$

$\sqrt[4]{1 - n^2} = 0.989229$

$1 - \sqrt[4]{1 - n^2} = 0.010771$

$\log(1 - \sqrt[4]{1 - n^2}) = 8.0322560$

subtract $\log n = 9.3136351$

$\log(\sin \lambda) = 8.7186209$

Therefore $\lambda = 2^\circ.59'.55''$

Thus the eccentric anomaly that corresponds to the maximum is $y = 3^s.2^\circ.59'.55''$.

Furthermore, to find the mean anomaly, we take $\log(\cos \lambda) = 9.9994050$ $\log n = 9.3136351$

subtract 4.6855749 from 9.3130401 and we have

4.6274652.

Therefore $n \cos \lambda = 42409''$, or $n \cos \lambda = 11^\circ.46'.49''$.

From the mean anomaly that corresponds to the maximum, being

$x = 90^\circ + \lambda + n \cos \lambda$, we will find $x = 3^s.14^\circ.46'.44''$, which we find agrees with the

maximum in the tables. Furthermore the true anomaly is $z = 90^\circ - \mu$, where

$\sin \mu = \dfrac{1 - \sqrt[4]{(1 - n^2)^3}}{n}$, from which we deduce the following calculation:

$\log \sqrt[4]{1-n^2}$ = 9.9952971

$\log \sqrt[4]{(1-n^2)^3}$ = 9.9859913

and $\sqrt[4]{(1-n^2)^3}$ = 0.9680356

$1-\sqrt[4]{(1-n^2)^3}$ = 0.0319643

$\log(1-\sqrt[4]{(1-n^2)^3})$ = 8.5046652

subtract $\log n$ = 9.3136351

$\log(\sin \mu)$ = 9.1910301

Therefore μ= 8°.55'.52"

Adding $\lambda + n\cos\lambda$ = 14°.46'.44"

The maximum is = 23°.42'.36", which does not differ even a second from the maximum represented in the (*astronomical*) tables, which verifies the previous theory. So as the semi-major axis of Mercury's orbit is = 38710 = *a*, we will have

$\log a$ = 4.5878232

$\log(\sqrt[4]{1-n^2})$ = 9.9952971

$\log r$ = 4.5831203

and *r* will be the distance from Mercury to the Sun where its equation is the greatest.

XX. Inversely, to find the eccentricity from the given maximum, I believed it necessary to place here the following table since this calculation can only be achieved by interpolations. In this table we find for each hundredth of eccentricity, the corresponding maximums, as well as eccentric and mean anomalies. The last column also provides the logarithm of the distance from the planet to the sun when the value of the equation is a maximum. In fact, calling this distance = *r*, and the semi-major axis = *a*, then we have $r = a\sqrt[4]{1-n^2}$. The last column contains the logarithms of the formula $\sqrt[4]{1-n^2}$, that being adding to the logarithms of the mean distance, will produce the logarithm of the desired distance *r*.

XXI. With the help of this table, if given any eccentricity, we will find by interpolation the corresponding maximum. Thus, the eccentricity of the Earth being = 0.0169, we find the eccentricities in the table.

	Eccentricity	Maximum
	0.0100	1°.8'.45"
	0.0200	2°.17'.31"
Difference	0.0100	1°.8'.46"

We subtract 0.0100 from the known value 0.0169 to get the difference 0.0069, and the result will be the following proportion:

100 : 1°.8'.46" = 69 : 47'.26".

In adding this angle 47'.26" to the smaller equation 1°.8'.45", we will have the maximum value of the equation of the center of the orbit of the Earth = 1°.56'.11".

	Eccentricity	Maximum
	0.090000	10°.19’.22”
	0.100000	11°.28’.20”
Difference	0.010000	1°.8’.58”

The eccentricity of Mars given in the tables is = 0.092988. We look for the two closest eccentricities with the maximums from our table. Now the eccentricity given exceeds the smaller value 0.090000 by 0.002998, and from this we calculate the proportion

10000 : 1°.8’.58” = 2998 : 20’.40”.

Add this angle 20’.40” to the preceding value of the equation (*in the table*), which is 10°.19’.22”, and we will get the maximum of the equation of the center of Mars’ orbit = 10°.40’.2”. This agrees perfectly with the (*astronomical*) tables.

XXII. The main use of the table below will be to determine the eccentricity when the maximum is known. Without this table, the problem is absolutely unsolvable. To demonstrate this by an example, we take the maximum of Mercury, which the (*astronomical*) tables show as 23°.42’.40”. We have already remarked that this is perfectly in agreement with the eccentricity provided by the same tables (*see section XIX.*) Let us take the two maximums that are the closest to this value:

	Maximum	Eccentricity
	23°.1’.32”	0.20
	24°.11’.19”	0.21
Difference	1°.9’.47”	0.01

Then let the smaller value of the equation be subtracted from the given value of Mercury's maximum:

23°.42'.40"
- 23°.1'.32"
0°.41'.8"

From this we find the proportion

1°.9'.47" : 0.01 = 0°.41'.8" : 0.0058944.

This number added to the smaller eccentricity 0.20, will give the eccentricity of Mercury's orbit = 0.2058944. This hardly differs from what has been observed, although we have assumed here that the value of the equation is 4" greater (*than that calculated in section XIX*). In fact, this addition of 4" only increases the logarithm of the eccentricity by 0.0000094, which was previously $\log n$ = 9.3136351.

XXIII. It is necessary to remark here that the two variables λ and μ from $\lambda + \mu + n\cos\lambda$ continually grow as the eccentricity n increases. The third term, $n\cos\lambda$, is of more importance; since its value disappears in the case $n = 0$ and also when $n = 1$. To find its maximum, we must examine the differential equation $dn\cos\lambda = nd\lambda\sin\lambda$. Now since $\sin\lambda = \frac{1-\sqrt[4]{1-n^2}}{n}$, or $n\sin\lambda = 1-\sqrt[4]{1-n^2}$, we will have $dn\sin\lambda + nd\lambda\cos\lambda = \frac{ndn}{2\sqrt[4]{1-n^2}}$. (***This is one of Euler's few misprints, this***

term should be $\frac{ndn}{2\sqrt[4]{(1-n^2)^3}}$.) We substitute here the preceding value $d\lambda = \frac{dn\cos\lambda}{n\sin\lambda}$,

and that will be $\frac{1}{\sin\lambda} = \frac{n}{2\sqrt[4]{(1-n^2)^3}}$, or $2\sqrt[4]{(1-n^2)^3} = 1 - \sqrt[4]{1-n^2}$. Let us suppose that $\sqrt[4]{1-n^2} = p$, and $2p^3$ will become $1-p$. This equation being solved by approximation, we will find that $\log p$ = 9.77067125, or p = 0.5897544, from there we find that n = 0.9375645, and the greatest value of $n\cos\lambda$ becomes = 48°.18'.10".40"'. Also, it is useful to observe that if the eccentricity is = 0.72388, the maximum value of the equation of the center will be exactly = 90°.

Excentricité. n	Plus grande equation $\lambda + \mu + n \cos\lambda$	Anomalie excentrique $3^s + \lambda$ λ	Anomalie moyenne $3^s+\lambda+n \cos\lambda$ $\lambda + n \cos\lambda$	Log. dist. au Soleil $l a + l \sqrt[4]{(1-nn)}$ $l \sqrt[4]{(1-nn)}$	$n \cos\lambda$	μ
0, 00	0, 0, 0	0, 0, 0	0, 0, 0	0,0000000	0, 0, 0	0, 0, 0
0, 01	1, 8,45	0, 8, 36	0, 42, 59	9,9999891	0, 34,23	0,25,47
0, 02	2,17,31	0, 17, 12	1, 25, 58	9,9999565	1, 8,46	0,51,34
0, 03	3,26,17	0, 25, 48	2, 8, 56	9,9999022	1,43, 8	1,17,22
0, 04	4,35, 4	0, 34, 24	2, 51, 54	9,9998261	2,17,30	1,43,10
0, 05	5,43,52	0, 43, 1	3, 34, 53	9,9997282	2,51,52	2, 8,59
0, 06	6,52,41	0, 51, 38	4, 17, 52	9,9996084	3,26,14	2,34,49
0, 07	8, 1,32	1, 0, 16	5, 0, 52	9,9994667	4, 0,36	3, 0,40
0, 08	9,10,26	1, 8, 55	5, 43, 53	9,9993029	4,34,57	3,26,33
0, 09	10,19,22	1, 17, 35	6, 26, 54	9,9991170	5, 9,19	3,52,28
0, 10	11,28,20	1, 26, 16	7, 9, 56	9,9989088	5,43,40	4,18,24
0, 11	12,37,21	1, 34, 59	7, 52, 59	9,9986782	6,18, 0	4,44,22
0, 12	13,46,26	1, 43, 43	8, 36, 3	9,9984252	6,52,20	5,10,23
0, 13	14,55,34	1, 52, 28	9, 19, 8	9,9981494	7,26,40	5,36,26
0, 14	16, 4,46	2, 1, 15	10, 2, 14	9,9978508	8, 0,59	6, 2,32
0, 15	17, 4, 1	2, 10, 3	10, 45, 20	9,9975292	8,35,17	6,28,41
0, 16	18,23,21	2, 18, 53	11, 28, 28	9,9971843	9, 9,35	6,54,53
0, 17	19,32,45	2, 27, 45	12, 11, 37	9,9968160	9,43,52	7,21, 8
0, 18	20,42,15	2, 36, 39	12, 54, 48	9,9964240	10,18, 9	7,47,27
0, 19	21,51,51	2, 45, 36	13, 38, 1	9,9960080	10,52,25	8,13,50
0, 20	23, 1,32	2, 54, 35	14, 21, 15	9,9955678	11,26,40	8,40,17
0, 21	24,11,19	3, 3, 37	15, 4, 31	9,9951031	12, 0,54	9, 6,48
0, 22	25,21,12	3, 12, 41	15, 47, 48	9,9946136	12,35, 7	9,33,24
0, 23	26,31,13	3, 21, 49	16, 31, 8	9,9942135	13, 9,19	10, 0, 5
0, 24	27,41,20	3, 31, 0	17, 14, 30	9,9935588	13,43,30	10,26,50
0, 25	28,51,35	3, 40, 14	17, 57, 54	9,9929928	14,17,40	10,53,41

n	$\lambda + \mu + n \cos \lambda$	λ	$\lambda + n \cos \lambda$	$l \sqrt[4]{(1-nn)}$	$n \cos \lambda$	μ
0, 25	28, 51, 35	3, 40, 14	17, 57, 54	9,9929928	14, 17, 40	10, 53, 41
0, 26	30, 1, 57	3, 49, 31	18, 41, 20	9,9924006	14, 51, 49	11, 20, 37
0, 27	31, 12, 28	3, 58, 52	19, 24, 49	9,9917816	15, 25, 57	11, 47, 40
0, 28	32, 23, 7	4, 8, 16	20, 8, 19	9,9911356	16, 0, 3	12, 14, 48
0, 29	33, 33, 57	4, 17, 45	20, 51, 53	9,9904620	16, 34, 8	12, 42, 4
0, 30	34, 44, 57	4, 27, 18	21, 35, 30	9,9897603	17, 8, 12	13, 9, 27
0, 31	35, 56, 6	4, 36, 55	22, 19, 9	9,9890301	17, 42, 14	13, 36, 56
0, 32	37, 7, 24	4, 46, 36	23, 2, 51	9,9882707	18, 16, 15	14, 4, 33
0, 33	38, 18, 55	4, 56, 22	23, 46, 36	9,9874816	18, 50, 14	14, 32, 19
0, 34	39, 30, 37	5, 6, 13	24, 30, 24	9,9866622	19, 24, 11	15, 0, 12
0, 35	40, 42, 30	5, 16, 9	25, 14, 16	9,9858118	19, 58, 7	15, 28, 14
0, 36	41, 54, 35	5, 26, 10	25, 58, 11	9,9849297	20, 32, 1	15, 56, 24
0, 37	43, 6, 53	5, 36, 17	26, 42, 10	9,9840153	21, 5, 53	16, 24, 43
0, 38	44, 19, 25	5, 46, 30	27, 26, 13	9,9830677	21, 39, 43	16, 53, 12
0, 39	45, 32, 12	5, 56, 50	28, 10, 20	9,9820861	22, 13, 30	17, 21, 52
0, 40	46, 45, 13	6, 7, 16	28, 54, 31	9,9810698	22, 47, 15	17, 50, 42
0, 41	47, 58, 28	6, 17, 48	29, 38, 46	6,9800178	23, 20, 58	18, 19, 42
0, 42	49, 12, 0	6, 28, 28	30, 23, 69	9,9789291	23, 54, 38	18, 48, 54
0, 43	50, 25, 49	6, 39, 15	31, 7, 31	9,9778027	24, 28, 16	19, 18, 18
0, 44	51, 39, 55	6, 50, 10	31, 52, 1	9,9766376	25, 1, 51	19, 47, 54
0, 45	52, 54, 19	7, 1, 12	32, 36, 35	9,9754327	25, 35, 22	20, 17, 43
0, 46	54, 9, 0	7, 12, 23	33, 21, 14	9,9741866	26, 8, 51	20, 47, 45
0, 47	55, 24, 2	7, 23, 43	34, 6, 0	9,9728983	26, 42, 17	21, 18, 2
0, 48	56, 39, 26	7, 35, 13	34, 50, 53	9,9715663	27, 15, 40	21, 48, 33
0, 49	57, 55, 10	7, 46, 52	35, 35, 51	9,9701891	27, 48, 59	22, 19, 19
0, 50	59, 11, 15	7, 58, 40	36, 20, 54	9,9687653	28, 22, 14	22, 50, 21

n	$\lambda + \mu + n \cos \lambda$	λ	$\lambda + n \cos \lambda$	$l \sqrt[4]{(1-nn)}$	$n \cos \lambda$	μ
0, 50	59, 11, 15	7, 58, 40	36, 20, 54	9,9687653	28, 22, 14	22, 50, 21
0, 51	60, 27, 44	8, 10, 39	37, 6, 4	9,9672932	28, 55, 25	23, 21, 40
0, 52	61, 44, 36	8, 22, 49	37, 51, 21	9,9657712	29, 28, 32	23, 53, 15
0, 53	63, 1, 56	8, 35, 12	38, 36, 47	9,9641973	30, 1, 35	24, 25, 9
0, 54	64, 19, 41	8, 37, 47	39, 12, 20	9,9625696	30, 34, 33	24, 57, 21
0, 55	65, 37, 52	8, 50, 34	39, 58, 0	9,9608860	31, 7, 26	25, 29, 52
0, 56	66, 56, 30	9, 13, 33	40, 53, 47	9,9591443	31, 40, 14	26, 2, 43
0, 57	68, 15, 42	9, 26, 49	41, 39, 46	9,9573420	32, 12, 57	26, 35, 56
0, 58	69, 35, 25	9, 40, 18	42, 25, 52	9,9554766	32, 45, 34	27, 9, 33
0, 59	70, 55, 43	9, 54, 2	43, 12, 6	9,9535452	33, 18, 4	27, 43, 37
0, 60	72, 16, 32	10, 8, 2	43, 58, 30	9,9515450	33, 50, 28	28, 18, 2
0, 61	73, 37, 58	10, 22, 20	44, 45, 5	9,9494726	34, 22, 45	28, 52, 53
0, 62	75, 0, 4	10, 36, 58	45, 31, 53	9,9473246	34, 54, 55	29, 28, 11
0, 63	76, 22, 51	10, 51, 55	46, 18, 52	9,9450973	35, 26, 57	30, 3, 59
0, 64	77, 46, 18	11, 7, 11	47, 6, 2	9,9427866	35, 58, 51	30, 40, 16
0, 65	79, 10, 28	11, 22, 49	47, 53, 25	9,9403880	36, 30, 36	31, 17, 3
0, 66	80, 35, 30	11, 38, 51	48, 41, 2	9,9378967	37, 2, 11	31, 54, 28
0, 67	82, 1, 18	11, 55, 16	49, 28, 53	9,9353076	37, 33, 37	32, 32, 25
0, 68	83, 27, 53	12, 12, 6	50, 16, 57	9,9326148	38, 4, 51	33, 10, 56
0, 69	84, 55, 28	12, 29, 25	51, 5, 19	9,9298121	38, 35, 54	33, 50, 9
0, 70	86, 24, 2	12, 47, 13	51, 53, 57	9,9268925	39, 6, 44	34, 30, 5
0, 71	87, 53, 37	13, 5, 32	52, 42, 53	9,9238485	39, 37, 21	35, 10, 44
0, 72	89, 24, 21	13, 24, 26	53, 32, 9	9,9206716	40, 7, 43	35, 52, 12
0, 73	90, 56, 15	13, 43, 56	54, 21, 45	9,9173525	40, 37, 49	36, 34, 30
0, 74	92, 29, 23	14, 4, 5	55, 11, 42	9,9138806	41, 7, 37	37, 17, 41
0, 75	94, 3, 53	14, 24, 55	56, 2, 3	9,9102445	41, 37, 8	38, 1, 50

n	$\lambda + \mu + n \cos \lambda$	λ	$\lambda + n \cos \lambda$	$l \sqrt[4]{(1 - nn)}$	$n \cos \lambda$	μ
0.75	94. 3.53	14.24.55	56. 2. 3	9.9102445	41.37. 8	38. 1.50
0.76	95.39.51	14.46.32	56.52.50	9.9064310	42. 6.18	38.47. 1
0.77	97.17.19	15. 8.57	57.44. 2	9.9024253	42.35. 5	39.33.17
0.78	98.56.26	15.32.15	58.35.41	9.8982107	43. 3.26	40.20.45
0.79	100.37.21	15.56.31	59.27.51	9.8937681	43.31.20	41. 9.30
0.80	102.20.17	16.21.53	60.20.39	9.8890756	43.58.46	41.59.38
0.81	104. 5.23	16.48.26	61.14. 4	9.8841080	44.25.38	42.51.19
0.82	105.52.41	17.16.16	62. 8. 7	9.8788360	44.51.51	43.44.40
0.83	107.42.42	17.45.33	63. 2.53	9.8732250	45.17.20	44.39.49
0.84	109.45.27	18.16.27	63.58.28	9.8672344	45.42. 1	45.46.59
0.85	111.31.31	18.49.14	64.55. 5	9.8608157	46. 5.51	46.36.26
0.86	113.31.59	19.24. 1	65.52.38	9.8539102	46.28.37	47.38.21
0.87	115.34.16	20. 1. 8	66.51.16	9.8464462	46.50. 8	48.43. 0
0.88	117.42.10	20.41. 1	67.51.14	9.8383348	47.10.13	49.50.56
0.89	119.55.28	21.24. 3	68.52.41	9.8294636	47.28.38	51. 2.47
0.90	122.14.47	22.10.54	69.55.53	9.8196884	47.44.59	52.18.54
0.91	124.41.47	23. 2.12	71. 1. 5	9.8088189	47.58.53	53.39.42
0.92	127.15.18	23.59. 7	72. 8.42	9.7965978	48. 9.35	55. 6.36
0.93	130. 0.21	25. 2.51	73.19.17	9.7826638	48.16.26	56.41. 4
0.94	132.59.48	26.15.49	74.33.48	9.7664882	48.17.59	58.25. 0
0.95	136.13.59	27.40.23	75.52.40	9.7472511	48.12.17	60.20.19
0.96	139.50.41	29.22.17	77.18.18	9.7235790	47.56. 1	62.32.23
0.97	144. 1.57	31.30.30	78.53.24	9.6928969	47.22.54	65. 8.33
0.98	149. 8.43	34.25. 1	80.44.15	9.6494238	46.19.14	68,24.28
0.99	156.10.30	39. 6.11	83. 7.14	9.5747133	44. 1. 3	73. 3.16
1.00	180 0. 0	90. 0 0	90. 0. 0	$-\infty$	0. 0. 0	90. 0. 0
9375645	132.13.33	25.56.55	74.15. 5⅔	9.7706712½	48.18.10⅔ maximum.	57.58.27⅔