

## Synopsis of Euler's paper

### **E105 -- Memoire sur la plus grande equation des planetes**

(Memoir on the Maximum value of an Equation of the Planets)

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#### **Preface**

The following summary of E 105 was constructed by abbreviating the collection of Notes. Thus, there is considerable repetition in these two items. We hope that the reader can profit by reading this synopsis before tackling Euler's paper itself.

#### **I. Planetary Motion as viewed from the earth vs the sun**

Euler discusses the fact that planets observed from the earth exhibit a very irregular motion. In general, they move from west to east along the ecliptic. At times however, the motion slows to a stop and the planet even appears to reverse direction and move from east to west. We call this retrograde motion. After some time the planet stops again and resumes its west to east journey.

However, if we observe the planet from the stand point of an observer on the sun, this retrograde motion will not occur, and only a west to east path of the planet is seen.

#### **II. The aphelion and the perihelion**

From the sun, (point  $O$  in figure 1) the planet (point  $P$ ) is seen to move on an elliptical orbit with the sun at one focus. When the planet is farthest from the sun, we say it is at the "aphelion" (point  $A$ ), and at the perihelion when it is closest. The time for the planet to move from aphelion to perihelion and back is called the period.

#### **III. Speed of planetary motion**

The planet's speed is slowest at the aphelion and fastest at the perihelion. The planet obeys Kepler's second law: "The radial line from the sun to the planet sweeps out equal areas in equal times."

#### IV. The fictitious planet which moves with constant speed

The more elliptic the orbit is, the greater is this variation in speed. If the orbit were a circle, the speed would always be constant and equal angles would be swept out in equal times.

We imagine a fictitious companion planet (point  $X$  in Figure 1) that circles the sun with the same period as our planet, but with uniform motion. Further, we assume that both the real and the fictitious planet reach the aphelion and perihelion points at the same time. As Euler says:

“After these two planets have passed by the aphelion, the false planet will appear to go faster than the true and the real planet will imperceptibly increase its speed until it will have caught the false one at the perihelion. Then it will pass its partner in speed, and will leave it behind until they rejoin again at the Aphelion.”

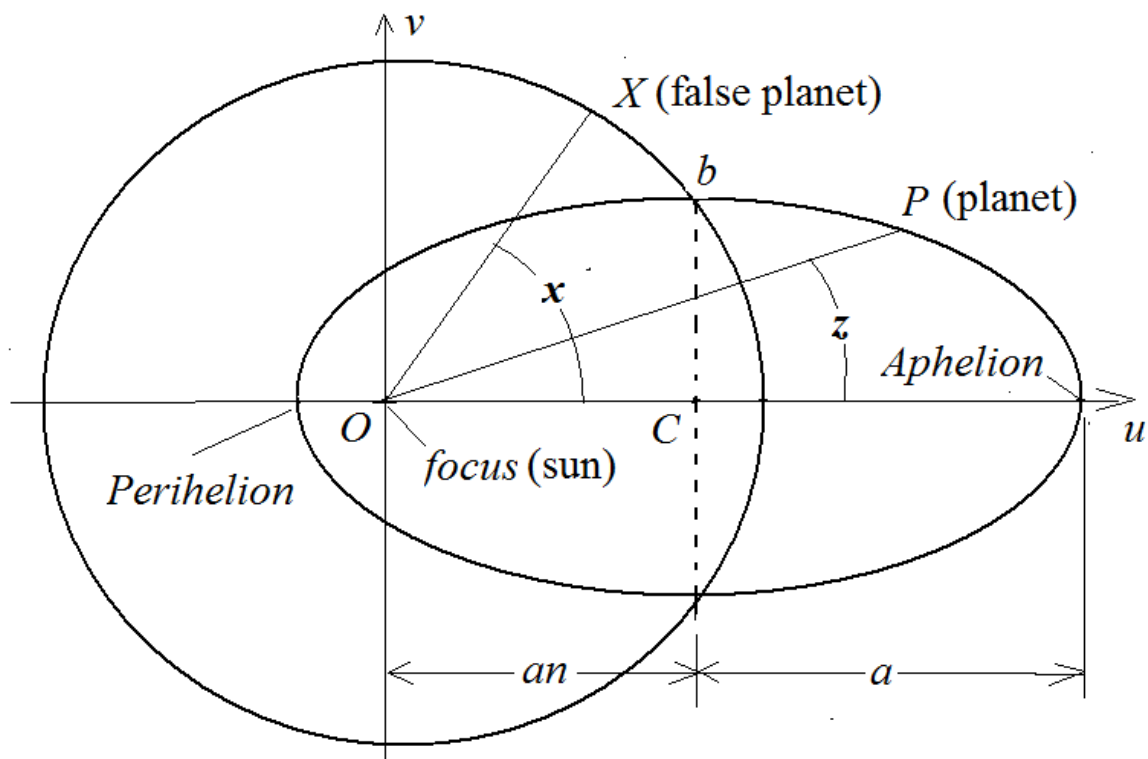


Figure 1: The planet  $P$  as observed from the sun at  $O$ ,

## V. The mean anomaly $x$ , the true anomaly $z$ and the “equation of the center”

Astronomers call the angle  $x$  made by the fictitious planet  $X$  the *mean anomaly*. The angle made by the true planet  $P$  is  $z$  and is called the *true anomaly*. The difference of these two angles is  $x - z$  and is called by astronomer’s the “equation of the center”.  $x - z$  is zero at the aphelion and gradually increases until it reaches a maximum near  $b$ , then it decreases to zero again at the perihelion.

## VI. The maximum of the equation of the center $x - z$

We will try to find the maximum of  $x - z$  and the value of the angle  $x$  at which this occurs. This maximum value must be a function of the eccentricity of the ellipse  $n$ . Euler notes that “And inversely, we will have to determine the eccentricity by the biggest equation.” This means that we will observe the maximum of  $x - z$ , and from this value, determine the eccentricity of the orbit. ***This is the main purpose of this paper, to determine the eccentricity of a planet by observations that have determined the maximum of  $x - z$ .***

The following sequence of five figures shows the progression of the planet  $P$  and the fictitious planet  $X$  as they move from the aphelion to the perihelion.

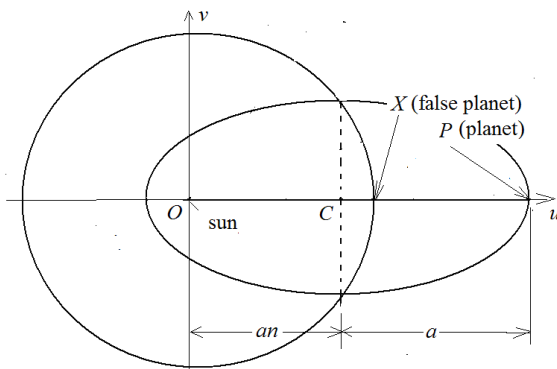


Figure 2a:  $x - z = 0$  at aphelion

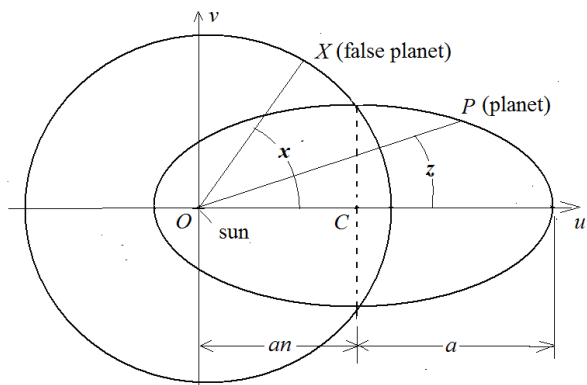


Figure 2b:  $x - z$  growing

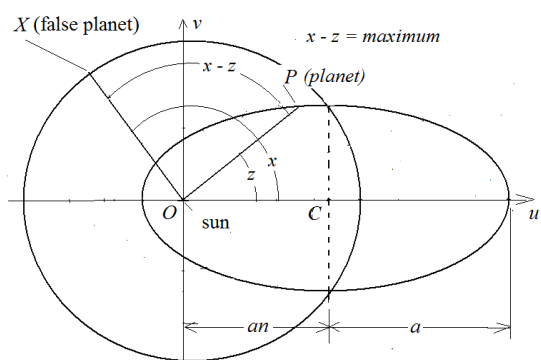


Figure 2c  $x - z$  is a maximum

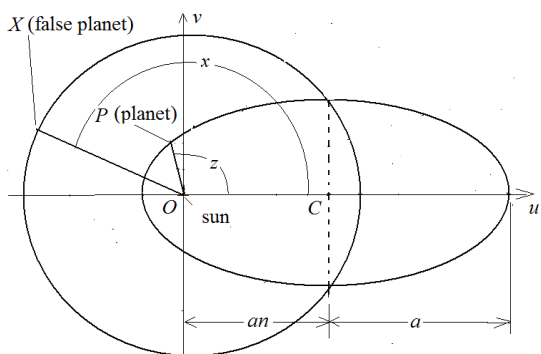


Figure 2d:  $x - z$  shrinking

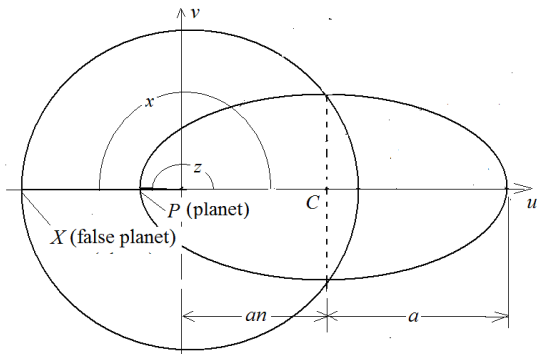


Figure 2e:  $x - z = 0$  at the perihelion

## VII. The focus and the eccentricity of the ellipse

Euler notes that this eccentricity equals the distance between the two foci of the ellipse divided by the length of the major axis. In Figure 3 we see that this is  $\frac{2an}{2a} = n$ .

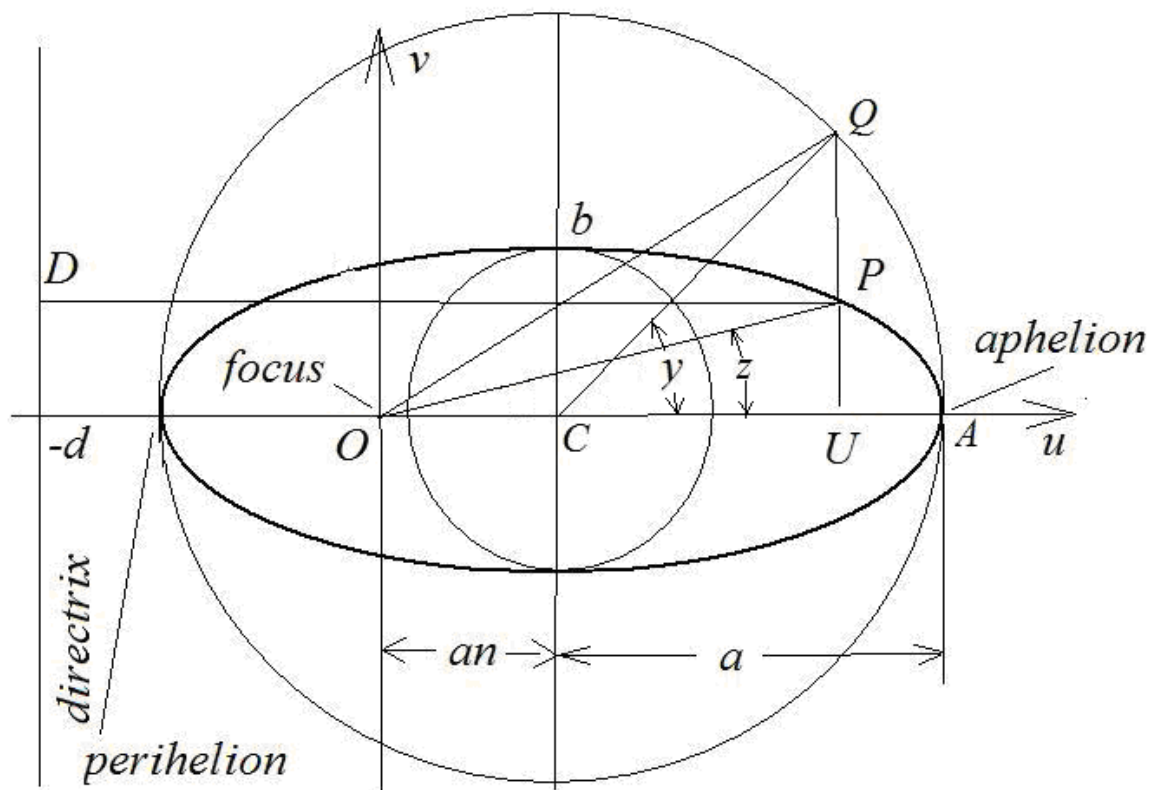


Figure 3: Features of the ellipse used in E105

When  $0 \leq n < 1$  the orbit is an ellipse, when  $n = 1$  it is a parabola, and when  $1 < n$  the orbit is a hyperbola. The distance from the sun to the aphelion is  $a + an$  and the distance from the sun to the perihelion is  $a - an$ . The length of the semi-minor axis is  $a\sqrt{1-n^2}$ .

### VIII. The eccentric anomaly $y$ , equations of the ellipse and Kepler's equation

Euler introduces the "eccentric anomaly"  $y$  which is shown in Figure 2. This angle  $y$  has the property that the equation of the ellipse traced by the planet at  $P$  can be written parametrically as  $u = an + a \cos y$  and  $v = b \sin y$ .

Euler gives without derivation the following equations

$$(A6) \quad r = a(1 + n \cos y).$$

$$(A7) \quad b = a\sqrt{1-n^2}.$$

$$(A8) \quad \cos z = \frac{n + \cos y}{1 + n \cos y}.$$

$$(A9) \quad \sin z = \frac{\sqrt{1-n^2} \sin y}{1 + n \cos y}$$

$$(A10) \quad \tan z = \frac{\sqrt{1-n^2} \sin y}{n + \cos y}$$

$$(A11) \quad x = y + n \sin y \text{ (Kepler's equation).}$$

### IX. Begin to find the equation of the center when $r = a$ . Finish in section XV.

Euler now wishes to examine closely the equation of the center  $x - z$ . In particular, he wishes to find the values of the angles  $x$  and  $z$  when  $r = a$ .

From (A6)  $r = a(1 + n \cos y)$ , we see that we need  $y = 90^\circ$ , and from (A11)  $x = y + n \sin y$ , we have  $x = y + n = \pi/2 + n$ .

From (A8) with  $y = 90^\circ$  we get  $\cos z = n$  and

$$(9.1) \quad z = \arccos n = 90^\circ - \arcsin n.$$

### X. The true anomaly $z$ in terms of the eccentric anomaly $y$ and the eccentricity $n$

Euler has previously obtained the relation

$$\begin{aligned}
 z = & y - n \sin y + \frac{1}{4} n^2 \sin 2y - \frac{1}{3 \cdot 4} n^3 (\sin 3y + 3 \sin y) + \\
 (10.1) \quad & \frac{1}{4 \cdot 8} n^4 (\sin 4y + 4 \sin 2y) - \frac{1}{5 \cdot 16} n^5 (\sin 5y + 5 \sin 3y + 10 \sin y) + \\
 & \frac{1}{6 \cdot 32} n^6 (\sin 6y + 6 \sin 4y + 15 \sin 2y) + \text{etc}
 \end{aligned}$$

**XI. Using calculus find when the maximum of  $x - z$  occurs in terms of  $n$  and  $y$ . See**

**(11.4). With  $y = \frac{\pi}{2} + \lambda$ , find  $\lambda$  in (11.5) and (11.6).**

Euler sets the problem:

“From the eccentricity  $n$  and the eccentric anomaly  $y$ , find the maximum of the equation”?

Starting with Kepler’s equation  $x = y + n \sin y$  and (A8)  $\cos z = \frac{n + \cos y}{1 + n \cos y}$ , Euler uses simple calculus to find

$$(11.3a) \quad 1 + n \cos y = \sqrt[4]{1 - n^2}$$

and

$$(11.4) \quad \cos y = \frac{\sqrt[4]{1 - n^2} - 1}{n}.$$

Note that this last result gives the exact value of  $y$  for which  $x - z$  is a maximum. Note that for small eccentricity  $n$ ,  $\cos y \approx -\frac{n}{4}$ , and so  $y \approx \frac{\pi}{2}$ .

Now we let  $\lambda$  be that small change in the angle by writing  $y = \frac{\pi}{2} + \lambda$  and thus using

$\sin \lambda = \sin (\pi/2 - y) = -\cos y$  and (11.4) we get

$$(11.5) \quad \sin \lambda = \frac{1 - \sqrt[4]{1 - n^2}}{n}$$

Thus knowing the eccentricity  $n$ , we can calculate  $\lambda$  from (11.5) and the eccentric anomaly from  $y = \frac{\pi}{2} + \lambda$ . Finally, the true anomaly  $z$  can be found from (10.1) or by inverting any of (A8), (A9) or (A10).

**XII. Determine more formulas for  $\lambda$  in terms of the eccentricity  $n$ .**

Euler finds

$$(12.3) \quad \sin \lambda = \frac{1}{4}n + \frac{1*3}{4*8}n^3 + \frac{1*3*7}{4*8*12}n^5 + \frac{1*3*7*11}{4*8*12*16}n^7 + \dots$$

and

$$(12.4) \quad \cos \lambda = 1 - \frac{1}{32}n^2 - \frac{49}{2048}n^4 - \frac{1233}{65536}n^6 - \dots$$

**XIII. Find the mean anomaly  $y$  directly in terms of  $n$  at the maximum of the equation. Get (13.1) through (13.3).**

Euler notes that having found  $\lambda$  from the previous section, we can now find  $y = 90^\circ + \lambda$ , then we can find from Kepler's Equation (A11)  $x = y + n \sin y$ , and  $z$  from (10.1) or any of (A8), (A9) or (A10). However, he would now like to find  $x$  and  $z$  directly from  $n$ .

So Euler begins with the problem: *Being given the eccentricity  $n$ , find the mean anomaly, to which corresponds the maximum of the equation.*

Without showing all the details of series manipulations Euler arrives at

$$(13.1) \quad x = 90^\circ + \frac{5}{4}n + \frac{25}{384}n^3 + \frac{1383}{40960}n^5 + \text{etc.}$$

Another finding is

$$(13.2) \quad x = 90^\circ + \lambda + n \cos \lambda, \text{ with}$$

$$(13.3) \quad \sin \lambda = \frac{1 - \sqrt[4]{1 - n^2}}{n}$$

**XIV/ Find the true anomaly  $z$  directly in terms of the eccentricity  $n$ . See (14.1) to (14.3).**

Euler now tries to find the true anomaly  $z$  from the eccentricity. He defines the new variable  $\mu$  through the equation

$$(14.1) \quad z = 90^\circ - \mu.$$

After several series manipulations which are not explained in detail he arrives at

$$(14.2) \quad \mu = \frac{3}{4}n + \frac{21}{128}n^3 + \frac{3409}{40960}n^5 + \text{etc.}$$

Euler also has

$$(14.3) \quad \sin \mu = \frac{1}{n} - \frac{1}{n} \sqrt[4]{(1 - n^2)^3}$$

See Figure 4 which illustrates these variables.



$$(16.1) \quad r = a\sqrt{1-n^2}. \text{ (Note that it is less than } a.)$$

If the value of  $x - z$  is called  $m$  and is given, it becomes very difficult to determine the eccentricity  $n$  from this. Euler states that we must use the equation  $m = \lambda + \mu + n \cos \lambda$  and try to determine  $n$  by substituting numbers for  $n$  and using trial and error to approximate the result by calculating values above and below  $m$ . In this way we can get bounds on a solution.

### XVII. Find the eccentricity $n$ as a series in powers of the maximum $m = x - z$ .

Euler now considers finding series for the eccentricity  $n$  in powers of the "greatest equation"  $m = x - z$ . These will be valuable when  $n$  is small. So he starts with (15.2)

$$m = 2n + \frac{11}{48}n^3 + \frac{599}{5120}n^5 + \dots$$

and inverts to get

$$(17.1) \quad n = \frac{1}{2}m - \frac{11}{768}m^3 - \frac{587}{2^{16} \cdot 15}m^5 - \dots$$

Euler reminds the reader that the value obtained from this equation must have 4.6855749 added to logarithm of the result to convert angles in seconds to radians. (See section XI.) The mean anomaly  $x$  can then be calculated from

$$(17.2) \quad x = 90^\circ + \frac{5}{8}m - \frac{5}{2^9 \cdot 3}m^3 - \frac{1}{2^9 \cdot 5}m^5 - \dots$$

Euler remarks that when  $n$  is small only the first term  $\frac{5}{8}m$  need be added to  $90^\circ$ .

### XVIII. A sample calculation for the planet Mercury. Find $x - z$ when the mean anomaly $y$ is 90 degrees.

In this section Euler does a numerical example of the use of the above results. He

chooses the planet Mercury which has an eccentricity of  $n = \frac{797}{3871} = 0.20589$ .

Now  $\log n = -0.686364849 = 9.31363515 - 10$ . He makes the approximation by

assuming that the maximum of the equation occurs where the eccentric anomaly  $y$  is

90 degrees. ( $\lambda = 0$ .) In this case, from (13.2) we get the mean anomaly  $x = 90^\circ + n$ .

Euler writes the result as  $x = 3^5.11'.47.48''$  where it appears that the symbol  $3'$

means 90 degrees. Using (9.1) Euler calculates  $z = 90^\circ - A \sin n$  and finds that  $A \sin n = 11^\circ.52'.54''$ . Thus  $x - z = 23^\circ.40'.42''$ , which is nearly two minutes less than the (known) maximum of the equation.

### **XIX. Calculate the maximum of $x - z$ for the planet Mercury**

Again we start with Mercury with the eccentricity  $n = \frac{797}{3871} = 0.20589$ . To find the maximum of the equation Euler begins using (11.5)  $\sin \lambda = \frac{1 - \sqrt[4]{1 - n^2}}{n}$  and using logarithms he finds  $\log(\sin \lambda) = 8.7186209$ . Thus  $\lambda = 2^\circ.59'.55''$  and the eccentric anomaly is  $y = 90^\circ 59' 55''$ .

From  $x = 90^\circ + \lambda + n \cos \lambda$  (Kepler's equation) Euler finds  $x = 104^\circ 46' 44''$ .

To find the true anomaly  $z$  Euler uses (14.3)  $\sin \mu = \frac{1 - \sqrt[4]{1 - n^2}}{n}$  and

obtains  $\mu = 8^\circ 55' 52''$ . Next Euler adds  $\lambda + n \cos \lambda = 14^\circ.46'.44''$  to obtain the maximum of the equation  $x - z = 23^\circ.42'.36''$ , which does not differ a second from the result found in the tables. Euler ends by finding the distance Mercury is from the sun when the maximum of the equation occurs. He obtains this from (16.1)

$$r = a \sqrt[4]{1 - n^2} \text{ with } 38710 = a.$$

### **XX. Euler explains his table**

The eccentricities  $n$  are given every hundredth in the first column, and the corresponding angle of the maximum of  $x - z$  is given in the second column. The last column also provides the logarithm of the distance from the planet to the sun, where its equation is the greatest.

### **XXI. Euler explains how to use linear interpolation to obtain the maximum of the equation from a given eccentricity.**

Euler uses simple linear interpolation for the planets Earth and Mars.

**XXII. Euler explains how to use linear interpolation to obtain the eccentricity when the maximum of the equation is given given.**

Euler uses the planet Mercury for a simple sample calculation.

**XXIII. Find the maximum of the term  $n \cos \lambda$  and mention the value of the eccentricity  $n$  when  $x - z = 90$  degrees.**

Euler notes that in our equation  $x - z = \lambda + \mu + n \cos \lambda$ , both  $\lambda$  and  $\mu$  increase as  $n$  increases, but this is not true for the term  $n \cos \lambda$ . In fact, this term is zero when  $n = 0$  and when  $n = 1$ . Euler then uses simple calculus to find when  $n \cos \lambda$  is a maximum and discovers that it occurs when the eccentricity is  $n = 0.9375645$ , and the actual maximum value is  $= 48^\circ.18'.10''.40'''$ . This last result is entered as the final line in Euler's table.

**Euler's Table**

Euler's table is meant to be used by astronomers who seek the eccentricity  $n$  of a planet from observations of the maximum of  $x - z$  (equation of the center). For this purpose only the first two columns of this seven column table are needed. The astronomer would find the maximum of  $x - z$  in column two and read off the eccentricity from column one.

