Notes on Euler's paper

E105 -- Memoire sur la plus grande equation des planetes

(Memoir on the Maximum value of an Equation of the Planets)

Compiled by
Thomas J Osler and Jasen Andrew Scaramazza
Mathematics Department
Rowan University
Glassboro, NJ 08028

Osler@rowan.edu

I. Planetary Motion as viewed from the earth vs the sun

` Euler discusses the fact that planets observed from the earth exhibit a very irregular motion. In general, they move from west to east along the ecliptic. At times however, the motion slows to a stop and the planet even appears to reverse direction and move from east to west. We call this retrograde motion. After some time the planet stops again and resumes its west to east journey.

However, if we observe the planet from the stand point of an observer on the sun, this retrograde motion will not occur, and only a west to east path of the planet is seen.

II. The aphelion and the perihelion

From the sun, (point O in figure 1) the planet (point P) is seen to move on an elliptical orbit with the sun at one focus. When the planet is farthest from the sun, we say it is at the "aphelion" (point A), and at the perihelion when it is closest. The time for the planet to move from aphelion to perihelion and back is called the period.

III. Speed of planetary motion

The planet's speed is slowest at the aphelion and fastest at the perihelion. The planet obeys Kepler's second law. The radial line from the sun to the planet sweeps out equal areas in equal times.

IV. The fictitious planet which moves with constant speed

The more elliptic the orbit is, the greater is this variation in speed. If the orbit were a circle, the speed would always be constant and equal angles would be swept out in equal times.

We imagine a fictitious companion planet (point X in Figure 1) that circles the sun with the same period as our planet, but with uniform motion. Further, we assume that both the real and the fictitious planet reach the aphelion and perihelion points at the same time. As Euler says:

"After these two planets have passed by the aphelion, the false planet will appear to go faster than the true and the real planet will imperceptibly increase its speed until it will have caught the false one at the perihelion. Then it will pass its partner in speed, and will leave it behind until they rejoin again at the Aphelion."

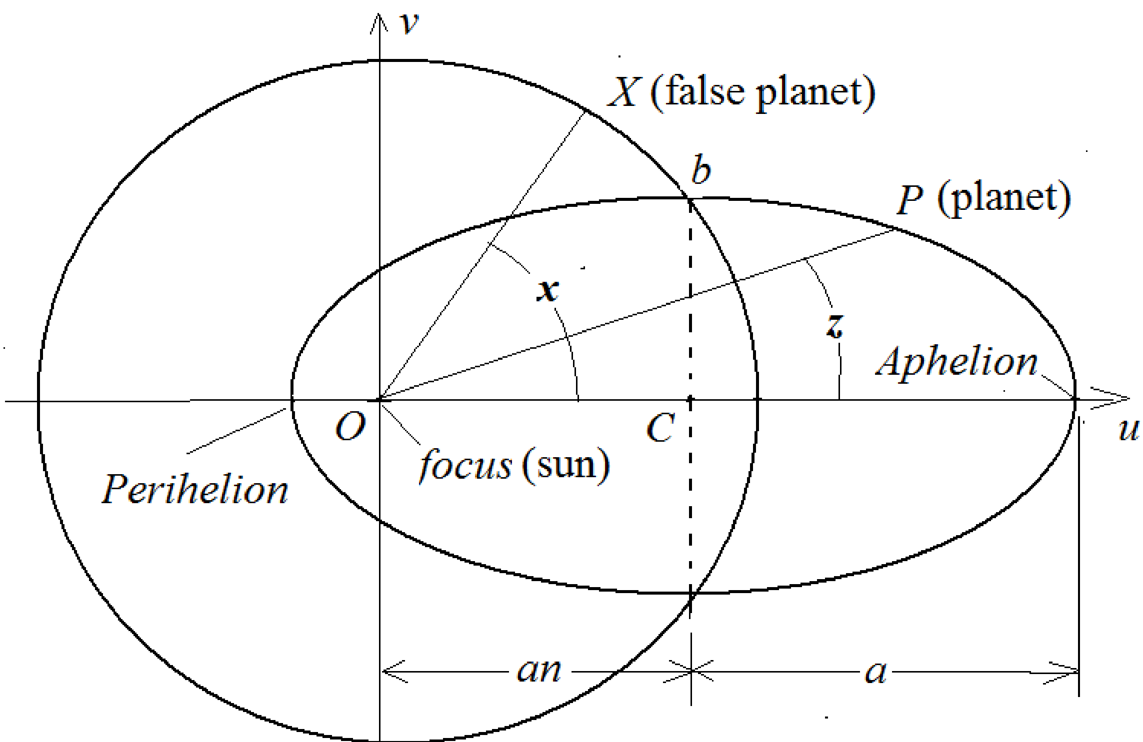

Figure 1: The planet *P* as observed from the sun at *O*,

V. The mean anomaly *x*, the true anomaly *z* and the "equation of the center"

Astronomers call the angle *x* made by the fictitious planet *X* the *mean anomaly.* The angle made by the true planet *P* is *z* and is called the *true anomaly.* The difference of these two angles is $x-z$ and is called by astronomer's the "equation of the center". $x-z$ is zero at the aphelion and gradually increases until it reaches a maximum near *b*, then it decreases to zero again at the perihelion.

VI. The maximum of the equation of the center $x-z$

We will try to find the maximum of $x-z$ and the value of the angle *x* at which this occurs. This maximum value must be a function of the eccentricity of the ellipse *n.* Euler notes that "And inversely, we will have to determine the eccentricity by the biggest equation." This means that we will observe the maximum of $x-z$, and from this value, determine the eccentricity of the orbit.

VII. The focus and the eccentricity on the ellipse

Euler notes that this eccentricity equals the distance between the two foci of the ellipse divided by the length of the major axis. In Figure 2 we see that this is $\frac{2an}{2a}=n$.

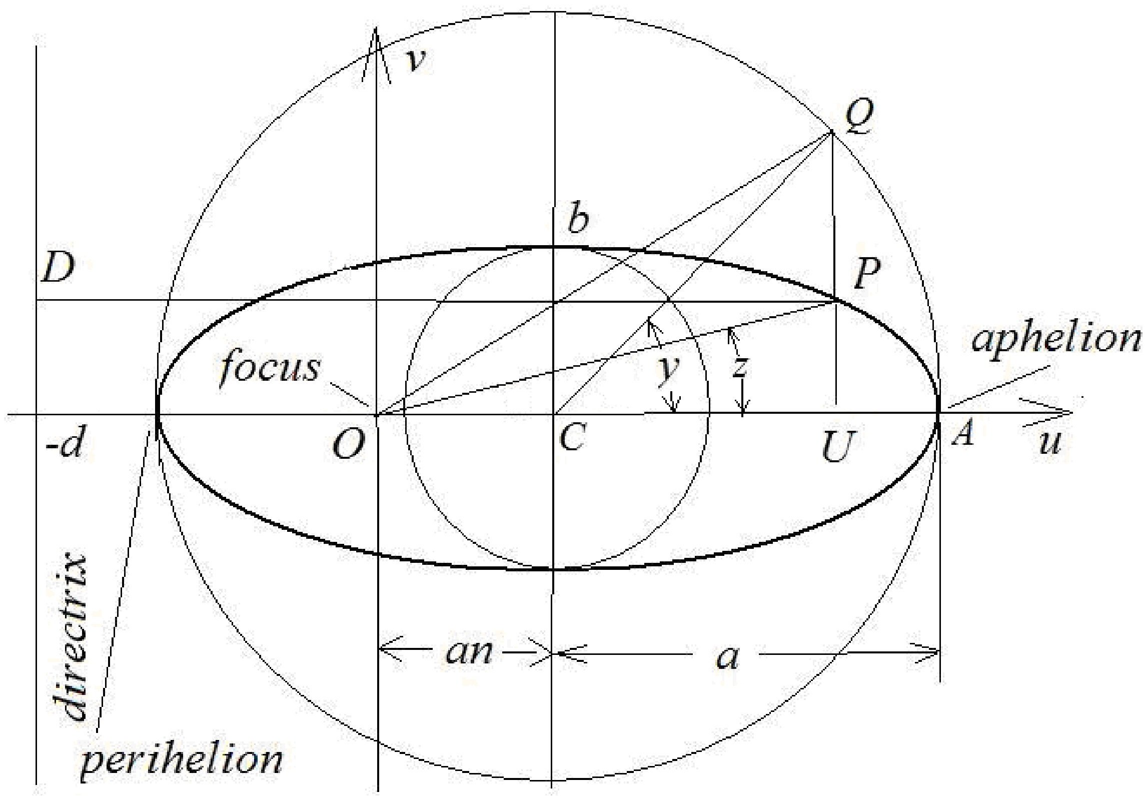

Figure 2: Features of the ellipse used in E105

When $0 \le n < 1$ the orbit is an ellipse, when $n = 1$ it is a parabola, and when $1 < n$ the orbit is a hyperbola. The distance from the sun to the aphelion is $a + an$ and the distance from the sun to the perihelion is $a - an$. The length of the semi-minor axis is $a\sqrt{1-n^2}$. See Appendix I for derivations of these features of the ellipse.

VIII. The eccentric anomaly *y* , equations of the ellipse and Kepler's equation

Euler introduces the "eccentric anomaly" *y* which is shown in Figure 2. This angle *y* has the property that the equation of the ellipse traced by the planet at *P* can be written parametrically as $u = an + a\cos y$ and $v = b\sin y$.

Euler gives without derivation the following equations

(A6) $r = a(1 + n\cos y)$.

(A7) $b = a\sqrt{1-n^2}$.

(A8) $\cos z = \dfrac{n + \cos y}{1 + n\cos y}$.

(A9) $\sin z = \dfrac{\sqrt{1-n^2}\sin y}{1 + n\cos y}$

(A10) $\tan z = \dfrac{\sqrt{1-n^2}\sin y}{n + \cos y}$

Equations (A6) through (A10) are derived below in Appendix I. He also lists Kepler's equation

(A11) $x = y + n\sin y$,

which is derived below in Appendix II.

IX. Begin to find the equation of the center when *r = a*. Finish in section XV.

Euler now wishes to examine closely the equation of the center $x - z$. In particular, he wishes to find the values of the angles *x* and *z* when $r = a$. From (A6) $r = a(1 + n\cos y)$, we see that we need $y = 90°$, and from (A11) $x = y + n\sin y$, we have $x = y + n = \pi/2 + n$. Here of course, the eccentricity *n* is an angle measured in radians. To convert an angle in seconds of arc to radians we must multiply by $\dfrac{\pi}{180 \times 3600}$. Since Euler does arithmetic with common logarithms we note that $\log\left(\dfrac{\pi}{180 \times 3600}\right) = -5.314425133$. Euler uses the value 4.6855749 which is

$10+\log\left(\frac{180}{\pi}\times 3600\right)=10-5.314425133\ \ =4.6855749$. (Probably Euler's log tables do not have negative values.)
From (A8) with $y=90°$ we get $\cos z=n$ and

(9.1) $z=\arccos n=90°-\arcsin n$.

X. The true anomaly *z* in terms of the eccentric anomaly *y* and the eccentricity *n*

Euler has previously obtained the relation

(10.1)
$$\begin{aligned} z &= y-n\sin y+\frac{1}{4}nn\sin 2y-\frac{1}{3\cdot 4}n^3(\sin 3y+3\sin y)+ \\ &\frac{1}{4\cdot 8}n^4(\sin 4y+4\sin 2y)-\frac{1}{5\cdot 16}n^5(\sin 5y+5\sin 3y+10\sin y)+ \\ &\frac{1}{6\cdot 32}n^6(\sin 6y+6\sin 4y+15\sin 2y)+etc \end{aligned}$$

See Appendix III for a Mathematica derivation of this result.

XI. Using calculus find when the maximum of *x – z* occurs in terms of *n* and *y*. See (11.4). With $y=\frac{\pi}{2}+\lambda$, find λ in (11.5) and (11.6).

Euler sets the problem:

"From the eccentricity *n* and the eccentric anomaly *y* , find the maximum of the equation"?

From Kepler's equation $x=y+n\sin y$ we have

(11.1) $dx=(1+n\cos y)dy$.

and differentiating (A8) $\cos z=\frac{n+\cos y}{1+n\cos y}$ we have

$$-\sin z\,dz=\frac{(1+n\cos y)(-\sin y)-(n+\cos y)(-n\sin y)}{(1+n\cos y)^2}dy$$

(11.2) $$\sin z dz=\frac{(1-n^2)\sin y}{(1+n\cos y)^2}dy$$

But by (A9) $\sin z=\frac{\sqrt{1-n^2}\sin y}{1+n\cos y}$, so (11.2) becomes

(11.3) $$dz = \frac{\sqrt{1-n^2}}{1+n\cos y}\,dy.$$

Since at the maximum of $x-z$, we have $dx = dz$, then from (11.1) and (11.3) we have

$1+n\cos y = \dfrac{\sqrt{1-n^2}}{1+n\cos y}$. Thus we have

(11.3a) $$1+n\cos y = \sqrt[4]{1-n^2}$$

and

(11.4) $$\cos y = \frac{\sqrt[4]{1-n^2}-1}{n}.$$

Note that this last result gives the exact value of y for which $x-z$ is a maximum. Note that for small eccentricity n, $\cos y \approx -\dfrac{n}{4}$, and so $y \approx \dfrac{\pi}{2}$.

Now we let λ be that small change in the angle by writing $y = \dfrac{\pi}{2}+\lambda$ and thus using $\sin\lambda = \sin(y-\pi/2) = -\cos y$ and (11.4) we get

(11.5) $$\sin\lambda = \frac{1-\sqrt[4]{1-n^2}}{n}$$

$$\sin\lambda = \frac{1-\sqrt[4]{1-n^2}}{n}\left(\frac{1+\sqrt[4]{1-n^2}}{1+\sqrt[4]{1-n^2}}\right)$$

$$\sin\lambda = \frac{1-\sqrt{1-n^2}}{n\left(1+\sqrt[4]{1-n^2}\right)}\left(\frac{1+\sqrt{1-n^2}}{1+\sqrt{1-n^2}}\right)$$

$$\sin\lambda = \frac{1-1+n^2}{n\left(1+\sqrt[4]{1-n^2}\right)\left(1+\sqrt{1-n^2}\right)}$$

(11.6) $$\sin\lambda = \frac{n}{\left(1+\sqrt[4]{1-n^2}\right)\left(1+\sqrt{1-n^2}\right)}.$$

Thus knowing the eccentricity *n*, we can calculate λ from (11.5) or (11.6) and the eccentric anomaly from $y = \dfrac{\pi}{2}+\lambda$. Finally, the true anomaly *z* can be found from (10.1) or by inverting any of (A8), (A9) or (A10).

XII. Determine formulas for λ in terms of the eccentricity *n*.

Let $n = \sin m$, and we now have $\sqrt{1-n^2} = \cos m$. It follows at once from (11.5) that

(12.1) $$\sin\lambda = \frac{1-\sqrt{\cos m}}{\sin m},$$

and from $\sin^2\lambda + \cos^2\lambda = 1$ that

(12.2) $$\cos\lambda = \frac{\sqrt{2\sqrt{\cos m} - \cos m - \cos m^2}}{\sin m}.$$

For small values of n the binomial theorem gives us

(12.2a) $$\sqrt[4]{1-n^2} = 1 - \frac{1}{4}n^2 - \frac{1*3}{4*8}n^4 - \frac{1*3*7}{4*8*12}n^6 - \frac{1*3*7*11}{4*8*12*16}n^8 + \cdots .$$

It now follows from (11.5) that

(12.3) $$\sin\lambda = \frac{1}{4}n + \frac{1*3}{4*8}n^3 + \frac{1*3*7}{4*8*12}n^5 + \frac{1*3*7*11}{4*8*12*16}n^7 + \cdots.$$

Euler also finds

(12.4) $$\cos\lambda = 1 - \frac{1}{32}n^2 - \frac{49}{2048}n^4 - \frac{1233}{65536}n^6 - \cdots,$$

without showing the details of series manipulations.

XIII. Find the mean anomaly *y* directly in terms of *n* at the maximum of the equation. Get (13.1) through (13.3).

Euler notes that having found λ from the previous section , we can now find $y = 90° + \lambda$, then we can find from Kepler's Equation (A11) $x = y + n\sin y$, and z from (10.1) or any of (A8), (A9) or (A10). However, he would now like to find x and z directly from n.

So Euler begins with the problem: *Being given the eccentricity n, find the mean anomaly, to which corresponds the maximum of the equation.*

Without showing all the details of series manipulations Euler arrives at

(13.1) $$x = 90^\circ + \frac{5}{4}n + \frac{25}{384}n^3 + \frac{1383}{40960}n^5 + etc.$$

From Kepler's equation $x = y - n\sin y$, and $y = \frac{\pi}{2} + \lambda$ we have

(13.2) $x = 90^\circ + \lambda + n\cos\lambda$, with (12.1)

(13.3) $$\sin\lambda = \frac{1-\sqrt[4]{1-n^2}}{n}.$$

XIV/ Find the true anomaly *z* directly in terms of the eccentricity *n*. See (14.1) to (14.3).

Euler now tries to find the true anomaly z from the eccentricity. He defines the new variable μ through the equation

(14.1) $$z = 90° - \mu.$$

After several series manipulations which are not explained in detail he arrives at

(14.2) $$\mu = \frac{3}{4}n + \frac{21}{128}n^3 + \frac{3409}{40960}n^5 + etc.$$

See Figure 3 which illustrates these variables.

From (A8) $\cos z = \dfrac{n+\cos y}{1+n\cos y}$ and (14.1) we get

$$\sin\mu = \frac{n-\sin\lambda}{1-n\sin\lambda}.$$

Replacing $\sin\lambda$ with (12.1) and simplifying we get

(14.3) $$\sin\mu == \frac{1}{n} - \frac{1}{n}\sqrt[4]{(1-n^2)^3}.$$

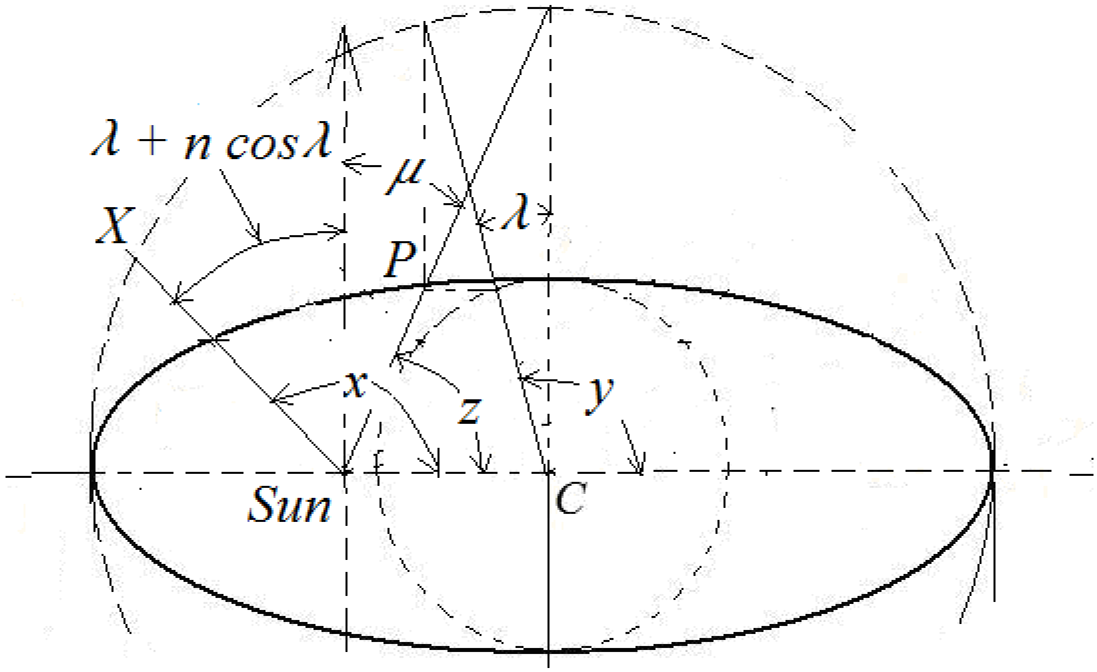

Figure 3

XV. Find the maximum of the equation $x - z$ directly in terms of the eccentricity *n*. See (15.1) through (15.3).

Euler now raises the question

Being given the eccentricity of the planet's orbit, find the greatest equation.

From (13.2) and (14.1) we get

(15.1) $$x - z = \lambda + \mu + n\cos\lambda \ ,$$

which can be expressed as

(15.2) $$x - z = 2n + \frac{11}{48}n^3 + \frac{599}{5120}n^5 + \cdots.$$

But when the distance from the planet to the sun is equal to half the major axis, the equation is

(15.3) $$x - z = n + \arcsin n = 2n + \frac{1}{6}n^3 + \frac{3}{40}n^5 + \cdots$$

Thus the maximum of $x - z$ surpasses this by a quantity = $\frac{1}{16}n^3 + \frac{43}{1024}n^5 + etc.$

XVI. Given the maximum of the equation $x - z$, determine the eccentricity *n*. This can only be done by numerical guessing.

From (A6) $r = a(1+n\cos y)$ and (11.3a) $1+n\cos y = \sqrt[4]{1-n^2}$ we have the distance from the sun to the planet at the maximum value of $x-z$ is

(16.1) $r = a\sqrt[4]{1-n^2}$. (Note that it is less than *a*.)

If the value of $x-z$ is called *m* and is given, it becomes very difficult to determine the eccentricity *n* from this. Euler states that we must use the equation $m = \lambda + \mu + n\cos\lambda$ and try to determine *n* by substituting numbers for *n* and using trial and error to approximate the result by calculating values above and below *m*. In this way we can get bounds on a solution.

XVII. Find the eccentricity *n* as a series in powers of the maximum *m = x – z.n*

Euler now considers finding series for the eccentricity *n* in powers of the "greatest equation" $m = x - z$. These will be valuable when *n* is small. So he starts with (15.2)

$$m = 2n + \frac{11}{48}n^3 + \frac{599}{5120}n^5 + \cdots$$

and inverts to get

(17.1) $$n = \frac{1}{2}m - \frac{11}{768}m^3 - \frac{587}{2^{16}\cdot 15}m^5 - \cdots.$$

Euler reminds the reader that the value obtained from this equation must have 4.6855749 added to logarithm of the result to convert angles in seconds to radians. (See section XI.) The mean anomaly *x* can then be calculated from

(17.2) $$x = 90° + \frac{5}{8}m - \frac{5}{2^9\cdot 3}m^3 - \frac{1}{2^9\cdot 5}m^5 - \cdots.$$

Euler remarks that when n is small only the first term $\frac{5}{8}m$ eed be added to $90°$.

(Euler probably used using (13.1) and (15.2) to obtain (17.2).)

XVIII. A sample calculation for the planet Mercury. Find *x – z* when the mean anomaly *y* is 90 degrees.

In this section Euler does a numerical example of the use of the above results. He

chooses the planet Mercury which has an eccentricity of $n = \frac{797}{3871} = 0.20589.$

Now $\log n = -0.686364849 = 9.31363515 - 10.$ He makes the approximation by assuming that the maximum of the equation occurs where the eccentric anomaly *y* is 90 degrees. ($\lambda = 0$.) In this case, from (13.2) we get the mean anomaly $x = 90° + n$. Euler writes the result as $x = 3^s.11°.47'.48''$ where it appears that the symbol 3' means 90 degrees. Using (9.1) Euler calculates $z = 90° - A\sin n$ and finds that $A\sin n = 11°.52'.54''$. Thus *x* – *z* = 23°.40'.42", which is nearly two minutes less than the (known) maximum of the equation.

XIX. Calculate the maximum of *x* - *z* for the planet Mercury

Again we start with Mercury with the eccentricity $n = \frac{797}{3871} = 0.20589.$ To find the maximum of the equation Euler begins using (11.5) $\sin\lambda = \frac{1-\sqrt[4]{1-n^2}}{n}$ and using logarithms he finds $\log(\sin\lambda) = 8.7186209$. Thus $\lambda = 2°.59'.55''$ and the eccentric anomaly is $y = 90°\ 59'\ 55''$.

From $x = 90° + \lambda + n\cos\lambda$ (Kepler's equation) Euler finds $x = 104°\ 46'\ 44''$.

To find the true anomaly *z* Euler uses (14.3) $\sin\mu = \frac{1-\sqrt[4]{(1-n^2)^3}}{n}$ and obtains $\mu = 8°\ 55'\ 52$. Next Euler adds $\lambda + n\cos\lambda$ = 14°.46'.44" to obtain the maximum of the equation *x* – *z* = 23°.42'.36", which does not differ a second from the result found in the tables. Euler ends by finding the distance Mercury is from the sun when the maximum of the equation occurs. He obtains this from (16.1)

$r = a\sqrt[4]{1-n^2}$ with 38710 = *a.*

XX. Euler explains his table

The eccentricities *n* are given every hundredth in the first column, and the corresponding angle of the maximum of *x* – *z* is given in the second column. The last column also provides the logarithm of the distance from the planet to the sun, where its equation is the greatest.

XXI. Euler explains how to use linear interpolation to obtain the maximum of the equation from a given eccentricity.

Euler uses simple linear interpolation for the planets Earth and Mars.

XXII. Euler explains how to use linear interpolation to obtain the eccentricity when the maximum of the equation is given given.

Euler uses the planet Mercury for a simple sample calculation.

XXIII. Find the maximum of the term *n* cos λ and mention the value of the eccentricity *n* when *x* – *z* = 90 degrees.

Euler notes that in our equation $x - z = \lambda + \mu + n\cos\lambda$, both λ and μ increase as n increases, but this is not true for the term $n\cos\lambda$. In fact, this term is zero when $n = 0$ and when $n = 1$. Euler then uses simple calculus to find when $n\cos\lambda$ is a maximum and discovers that it occurs when the eccentricity is n = 0.9375645, and the actual maximum value is = 48°.18'.10".40"'. This last result is entered as the final line in Euler's table.

Euler's Table

Euler's table is meant to be used by astronomers who seek the eccentricity n of a planet from observations of the maximum of $x - z$ (equation of the center). For this purpose only the first two columns of this seven column table are needed. The astronomer would find the maximum of $x-z$ in column two and read off the eccentricity from column one.

Appendix I: Features of the ellipse

In Figure 2 we see the ellipse with focus at the origin *O* of the *uv*-plane. We imagine that the sun is at point *O* and the planet is at point *P*. Following Euler, we will use the variables:

n = eccentricity of the ellipse
z = true anomaly, (usually the polar angle θ)
y = eccentric anomaly, (often E is used)
a = semi-major axis
b = semi-minor axis.
$r = OP$ (usual polar radius)
t = time for planet to move from A to P
T = period of the planet
$\omega = \frac{2\pi}{T}$ = mean angular velocity
$x = \omega t$ = mean anomaly

Definition:

Let *O* be a fixed point (focus) and $u=-d$ be a fixed line (directrix). The locus of all points *P* such that the ratio of the distance from the focus to the distance from the directrix is a constant *n* (eccentricity) is called a conic section. Thus we have

$$\frac{OP}{DP} = n \quad \text{(a constant).}$$

If $0<n<1$ the curve is an ellipse. If $n=1$ the curve is a parabola. If $n>1$ the curve is a hyperbola.

Since $DP = d + r\cos z$ we have

$$\frac{OP}{DP} = \frac{r}{d + r\cos z} = n \ ,$$

and solving for *r* we get

$$\text{(A1)} \qquad r = \frac{dn}{1 - n\cos z} \ .$$

At the perihelion we have from the definition $\frac{r}{d-r} = n$, and solving for *r* we get

(A2) $$r_{perihilion} = \frac{nd}{1+n}.$$

At the aphelion we have $\frac{r}{d+r} = n$ so we get

(A3) $$r_{aphelion} = \frac{nd}{1-n}.$$

Since the major axis of our ellipse is

$$2a = r_{perihilion} + r_{aphelion} = \frac{nd}{1+n} + \frac{nd}{1-n} = \frac{2nd}{1-n^2},$$

Thus it follows that

$$nd = a(1-n^2),$$

and so our equation for the ellipse (A1) becomes

(A4) $$r = \frac{a(1-n^2)}{1-n\cos z}.$$

Returning to (A2) we now have

$$r_{perihilion} = \frac{nd}{1+n} = \frac{a(1-n^2)}{1+n} = a - an.$$

This last relation tells us that the distance from the focus to the center of the ellipse is an as shown on the figure.

From Figure 2 we can now calculate the distance *OU*.

(A5) $$OU = r\cos z = an + a\cos y.$$

We can rewrite (A4) as

$$r - nr\cos z = a(1-n^2),$$

and using (A5) to eliminate $r\cos z$ we get

$$r - n(an + a\cos y) = a(1-n^2)$$

which simplifies to

(A6) $$r = a(1 + n\cos y).$$

From this we infer that when $y = \pi/2$, then $r = a$. It follows then from Figure 2 that we have a right triangle *OCb* with legs *an*, and *b* and hypotenuse *a*. It follows that the semi minor axis is given by

(A7) $$b = a\sqrt{1-n^2}\ .$$

Notice also that

$$\cos z = \frac{OU}{OP} = \frac{an + a\cos y}{r},$$

and replacing *r* by (6) we get

(A8) $$\cos z = \frac{n + \cos y}{1 + n\cos y}\,.$$

Also

$$\sin z = \frac{PU}{OP} = \frac{b\sin y}{r},$$

and using (A6) and (A7) this becomes

(A9) $$\sin z = \frac{\sqrt{1-n^2}\,\sin y}{1 + n\cos y}$$

From (A8) and (A9) we get

(A10) $$\tan z = \frac{\sqrt{1-n^2}\,\sin y}{n + \cos y}$$

Equations (A6) through (A10) appear in the last few lines of section VIII of E105.

Appendix II: Derivation of Kepler's equation

Kepler's equation is

(A11) $$x = y + n\sin y\,.$$

where $x = \omega t$ is the mean anomaly. This equation is the mathematical statement of Kepler's second law of planetary motion:

"The planet sweeps out equal areas in equal times."

Kepler's equation (A11) appears near the end of section VIII. (To see Kepler's original derivation, which is much like ours see to [1].)

Referring to Figure 2 we see that this translates to

(A12) $$\frac{Area\ OAP}{Area\ of\ full\ ellipse} = \frac{t}{T}.$$

Now

$$Area\ OAP = \frac{b}{a}\ Area\ OAQ$$

$$Area\ OAP = \frac{b}{a}\ (Area\ Triangle\ OCQ + Area\ Sector\ CAQ)$$

$$Area\ OAP = \frac{b}{a}\ (\frac{1}{2}(an)a\sin y + \frac{1}{2}a^2 y)$$

(A13) $$Area\ OAP = \frac{1}{2}ab(n\sin y + y).$$

Using (A13) to simplify (A12) we get

$$\frac{\frac{1}{2}ab(n\sin y + y)}{\pi ab} = \frac{t}{T}$$

$$y + n\sin y = \frac{2\pi t}{T} = \omega t = x$$

This is Kepler's equation.

Appendix III: Mathematica verification of Euler's equation relating the true anomaly *z* to the eccentric anomaly *y* in powers of the eccentricity *n*.

Euler has previously obtained the relation

$$z = y - n\sin y + \frac{1}{4}nn\sin 2y - \frac{1}{3\cdot 4}n^3(\sin 3y + 3\sin y) + \frac{1}{4\cdot 8}n^4(\sin 4y + 4\sin 2y) - \frac{1}{5\cdot 16}n^5(\sin 5y + 5\sin 3y + 10\sin y) + \frac{1}{6\cdot 32}n^6(\sin 6y + 6\sin 4y + 15\sin 2y) + etc$$

We used Mathematica to verify this result. Starting with (A8) we can write

$$z = \cos^{-1}\left(\frac{n + \cos y}{1 + n\cos y}\right).$$

Euler's relation is the Taylor's series in powers of *n* of this expression. The Mathematica code is

```
z[y_,n_]:=ArcCos[(n+Cos[y])/(1+n*Cos[y])]
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```
Series[z[y,n],{n,0,5}]
```

$$\text{ArcCos[Cos[y]]}-\sqrt{\text{Sin[y]}^2}\ \text{n}+1/2\ \text{Cos[y]}\ \sqrt{\text{Sin[y]}^2}\ \text{n}^2-1/6\ ((2+\text{Cos[2 y]})\ \sqrt{\text{Sin[y]}^2})\ \text{n}^3+1/16\ (5\ \text{Cos[y]}+\text{Cos[3 y]})\ \sqrt{\text{Sin[y]}^2}\ \text{n}^4-1/40\ ((8+6\ \text{Cos[2 y]}+\text{Cos[4 y]})\ \sqrt{\text{Sin[y]}^2})\ \text{n}^5+\text{O[n]}^6$$

This simplifies to

$$z = y - \sin(y)n + \frac{1}{2}\cos y \sin y\ n^2 - \frac{1}{6}\left((2+\cos 2y)\sin y\right)n^3 + \frac{1}{16}\left(5\cos y + \cos 3y\right)\sin y\ n^4 - \frac{1}{40}\left(8 + 6\cos 2y + \cos 4y\right)\sin y\ n^5 + On^6$$

Subtracting the above from Kepler's equation $x = y + n\sin y$ we get

$$x - z = 2\sin(y)n - \frac{1}{2}\cos y \sin y\ n^2 + \frac{1}{6}\left((2+\cos 2y)\sin y\right)n^3 - \frac{1}{16}\left(5\cos y + \cos 3y\right)\sin y\ n^4 + \frac{1}{40}\left(8 + 6\cos 2y + \cos 4y\right)\sin y\ n^5 + On^6$$

Differentiating to find the maximum we get

$$\frac{d(x-z)}{dy} = 2\cos(y)n - \frac{1}{2}(-\sin y \sin y + \cos y \cos y)\ n^2 + \frac{1}{6}\left((-2\sin 2y)\sin y + (2+\cos 2y)\cos y\right)n^3 - \frac{1}{16}\left(\left(-5\sin y - 3\sin 3y\right)\sin y + (...)\cos y\right)n^4 + \frac{1}{40}\left(\left(-12\sin 2y - 4\sin 4y\right)\sin y + (...)\cos y\right)n^5 + On^6$$

If we set $y = \pi/2$ we get

$$\left.\frac{d(x-z)}{dy}\right|_{y=\pi/2} = \frac{1}{2}n^2 + \frac{1}{8}n^4 + On^6$$

so $y = \pi/2$ is not at the maximum of $x - z$, but is close to the maximum.

References

[1] Kepler, Johannes, *Epitome of Copernican Astronomy and Harmonies of the World*, (Translated by Charles Glenn Wallis), Prometheus Books, New York, 1995, p. 152.