

CAPUT XVIII.

DE USU CALCULI DIFFERENTIALIS
IN RESOLUTIONE FRACTIONUM.

403.

Methodus fractionem quamvis propositam in fractionibus simplicibus resolvendi, quam in Introductione exposuimus, etsi per se satis est facilis; tamen ope calculi differentialis ita perfici potest, ut saepenumero multo minori negotio in usum vocari possit. Praecipue vero si denominator fractionis resolvendae fuerit indefiniti gradus, methodus ante exposita plerumque non mediocriter impeditur, dum loco quantitatis incognitae substitutio valoris, quem ex quopiam factore induit, fieri debet. Imprimis autem his casibus divisio denominatoris per factorem iam inventum nimis fit molesta. Quae operatio, si calculus differentialis in subsidium vocetur, evitari poterit, ita ut non opus sit alterum denominatoris factorem, qui oritur, si denominator per factorem iam cognitum dividatur, nosse. Hunc autem usum praestat methodus determinandi valorem fractionis, cuius numerator ac denominator certo casu ambo evanescent, cuius beneficio, quemadmodum resolutio fractionum iam supra tradita commodior & tractabilior reddi queat, hoc Capite doceamus, simulque finem huic libro, in quo usum calculi differentialis in Analyfi exposuimus, imponamus.

404. Si igitur proposita fuerit fractio quaecunque $\frac{P}{Q}$; cuius numerator ac denominator sint functiones variabilis quantitatis x , rationales & integrae; primum videndum est, utrum x in numeratore P tot pluresve dimensiones habeat, quam in deno-

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iufmodi $ff - 2fgx \cos \phi + ggxx$, ex eo orietur fractio fim-
 plex talis formae $\frac{u + ax}{ff - 2fgx \cos \phi + ggxx}$; & , si duo. hu-
 iufmodi factores fuerint aequales uti $(ff - 2fgx \cos \phi + ggxx)^2$,
 hinc prodibunt duae fractiones

$$\frac{u + ax}{(ff - 2fgx \cos \phi + ggxx)^2} + \frac{v + bx}{ff - 2fgx \cos \phi + ggxx}$$

Huiufmodi autem factor cubicus $(ff - 2fgx \cos \phi + ggxx)^3$
 dabit tres fractiones fim-
 plices, biquadratus quatuor, & ita porro.

406. Resolutio ergo fractionis cuiuscunque $\frac{P}{Q}$ ita infli-
 tuatur. Quaerantur primo omnes factores tam fim-
 plices seu binomiales, quam trinomiales denominatoris Q , & si qui
 fuerint inter fe aequales, ii probe noentur, & instar unius
 habeantur. Tum ex fingulis his denominatoris factoribus eli-
 ciantur fractiones fim- plices, vel modo iam supra ostenso, vel
 eo, quem hic sumus tradituri, & qui pro lubitu in locum
 prioris substitui poterit. Quo facto aggregatum omnium ista-
 rum fractionum simplicium una cum parte integra, si quam
 continet fractio proposita $\frac{P}{Q}$, huius valorem exhaurient. In-
 ventioem quidem factorum denominatoris Q hic tanquam
 cognitam assumimus, cum pendeat a resolutione aequationis
 $Q = 0$; methodumque hic trademus per calculum differen-
 tiale pro dato quovis denominatoris factore fractionem fim-
 plicem inde ortam definiendi. Quod, cum istarum fractionum
 simplicium denominatores iam habeantur, praestabitur, si nu-
 meratorem cuiusque fractionis investigare doceamus.

407. Ponamus ergo fractionis $\frac{P}{Q}$ denominatorem Q fa-
 ctorem habere $f + gx$, ita ut sit $Q = (f + gx)S$ neque vero
 hic alter factor S insuper eundem factorem $f + gx$ contineat.
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simplex $\frac{U}{f+gx}$, existente $U = \frac{gPdx}{dQ}$, postquam hic ubique loco x valor $\frac{-f}{g}$ ex aequatione $f+gx=0$ oriundus fuerit substitutus. Hoc ergo modo non necesse est, ut ante quae- ratur alter denominatoris Q factor S , qui oritur, si Q per $f+gx$ dividatur. Hinc, si Q non in factoribus exprimitur, hanc divisionem saepe non parum molestam, praecipue si in denominatore Q habeat exponentes indefinitos omittere poterimus, cum valor ipsius U ex formula $\frac{gPdx}{dQ}$ obtineatur. Sin autem denominator Q iam in factoribus fuerit expressus, ita ut inde valor ipsius S sponte pateat, tum praefenda erit altera expressio, qua invenimus $U = \frac{P}{S}$, ponendo pariter ubique $x = \frac{-f}{g}$. Sicque pro inveniend. valore ipsius U quovis casu ea formula adhiberi poterit, quae commodior & expeditior videatur. Usum autem novae formulae aliquot exemplis illustrabimus.

EXEMPLUM I.

Sit proposita ista fractio $\frac{x^2}{1+x^{17}}$, cuius fractionem simplicem ex denominatoris factore $1+x$ oriundam definiri oporteat.

Quoniam hic est $Q = 1+x^{17}$, cuius etfi factor $1+x$ constat, tamen si, uti prima methodus postulat, per eum dividere velimus, prodiret

$$S = 1 - x + xx - x^3 + \dots + x^{16}$$

Commodius igitur, utemur nova formula $U = \frac{gPdx}{dQ}$; quia itaque est $f=1$, $g=1$, & $P=x^2$, ob $dQ = 17x^{16}dx$ fiet

fiet $\mathcal{U} = \frac{x^9}{17x^{16}} = \frac{1}{17x^7}$, posito $x = -1$, unde fit
 $\mathcal{U} = -\frac{1}{17}$, & fractio simplex ex denominatoris factore $1+x$

oriunda erit $\frac{-1}{17(1+x)}$.

E X E M P L U M II.

Proposita fractione $\frac{x^m}{1-x^{2n}}$ fractionem simplicem ex denomi-
 natoris factore $1-x$ oriundam investigare.

Ob factorem propositum $1-x$, erit $f=1$, & $g=-1$.
 Tum vero denominator $Q=1-x^{2n}$ dat $dQ=-2nx^{2n-1}dx$;
 unde propter $P=x^m$ obtinebitur $\mathcal{U} = \frac{-x^m}{-2nx^{2n-1}}$. Positoque ex

aequatione $1-x=0$, $x=1$, fiet $\mathcal{U} = \frac{1}{2n}$; ita ut fractio
 simplex futura sit haec $\frac{1}{2n(1-x)}$.

E X E M P L U M III.

Proposita fractione $\frac{x^m}{1-4x^k+3x^n}$, eius fractionem simplicem
 ex denominatoris factore $1-x$ oriundam determinare.

Hic ergo fit $f=1$; $g=-1$; $P=x^m$; $Q=1-4x^k+3x^n$
 & $\frac{dQ}{dx} = -4kx^{k-1} + 3nx^{n-1}$; unde fit $\mathcal{U} = \frac{-x^m}{-4kx^{k-1} + 3nx^{n-1}}$

& posito $x=1$, erit $\mathcal{U} = \frac{1}{4k-3n}$. Fractio ergo simplex
 ex isto denominatoris factore simplici $1-x$ oriunda erit
 $\frac{1}{(4k-3n)(1-x)}$.

409. Ponamus nunc fractionis $\frac{P}{Q}$ denominatorem Q factorem habere quadratum $(f + gx)^2$, & fractiones simplices hinc oriundas esse $= \frac{U}{(f + gx)^2} + \frac{V}{f + gx}$. Sit $Q = (f + gx)^2 S$ & complementum $= \frac{V}{S}$; ita ut fit $\frac{P}{S} = \frac{U}{Q} + \frac{V}{(f + gx)^2} + \frac{V}{S}$; & $V = \frac{P - US - V(f + gx)S}{(f + gx)^2}$. Quia nunc V est functio integra, necesse est ut fit $P - US - V(f + gx)S$ divisible per $(f + gx)^2$; & cum S factorem $f + gx$ amplius non contineat, quoque haec expressio $\frac{P}{S} - U - V(f + gx)$ divisible erit per $(f + gx)^2$; ideoque factio $f + gx = 0$, seu $x = \frac{-f}{g}$ non solum ipsa, sed etiam eius differentiale $d. \frac{P}{S} - Vgd x$ evanescet. Fiat ergo $x = \frac{-f}{g}$, eritque ex priori aequatione $U = \frac{P}{S}$; ex posteriori vero erit $V = \frac{1}{gd x} d. \frac{P}{S}$; quibus valoribus inventis habebuntur fractiones quaesitae: $\frac{U}{(f + gx)^2} + \frac{V}{f + gx}$.

EXEMPLUM.

Proposita fractione $\frac{x^m}{1 - 4x^3 + 3x^4}$ cuius denominator factorem habet $(1 - x)^2$, invenire fractiones simplices hinc oriundas; Cum hic fit $f = 1, g = -1, P = x^m$ & $Q = 1 - 4x^3 + 3x^4$, erit $S = 1 + 2x + 3xx$; $\frac{P}{S} = \frac{x^m}{1 + 2x + 3xx}$; &

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Ex prima aequatione ergo erit $\mathcal{A} = \frac{P}{S}$

Ex secunda vero erit $\mathcal{B} = \frac{1}{g dx} d. \frac{P}{S}$

Ex tertia denique definitur $\mathcal{C} = \frac{1}{2g^2 dx^2} dd. \frac{P}{S}$

411. Generaliter ergo si fractionis $\frac{P}{S}$ denominator Q factorem habeat $(f+gx)^n$, ita ut sit $Q = (f+gx)^n S$; positis fractionibus simplicibus ex hoc factoris $(f+gx)^n$ oriendis his: $\frac{\mathcal{A}}{(f+gx)^n} + \frac{\mathcal{B}}{(f+gx)^{n-1}} + \frac{\mathcal{C}}{(f+gx)^{n-2}} + \frac{\mathcal{D}}{(f+gx)^{n-3}} + \frac{\mathcal{E}}{(f+gx)^{n-4}} + \&c.$ quoad ad ultimam, cuius denominator est $f+gx$, perveniatur, si ratiocinium ut ante instituitur, reperietur haec expressio: $\frac{P}{S} - \mathcal{A} - \mathcal{B}(f+gx) - \mathcal{C}(f+gx)^2 - \mathcal{D}(f+gx)^3 - \mathcal{E}(f+gx)^4 - \&c.$ divisibilis esse debere per $(f+gx)^n$, hinc tam plura quam singula eius differentialia usque ad gradum $n-1$, casu $x = -\frac{f}{g}$ evanescere debent. Ex quibus aequationibus concludetur fore ponendo ubique $x = -\frac{f}{g}$

$$\mathcal{A} = \frac{P}{S}$$

$$\mathcal{B} = \frac{1}{1g dx} d. \frac{P}{S}$$

$$\mathcal{C} = \frac{1}{1.2g^2 dx^2} dd. \frac{P}{S}$$

$$\mathcal{D} = \frac{1}{1.2.3g^3 dx^3} d^3. \frac{P}{S}$$

$$\mathcal{E} = \frac{1}{1.2.3.4g^4 dx^4} d^4. \frac{P}{S} \quad \&c.$$

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Ponamus esse $Q = (ff - 2fgx \cos \phi + ggxx)S$, atque S præterea per $ff - 2fgx \cos \phi + ggxx$ non esse divisibile. Sit fractio ex isto factore denominatoris oriunda:

$$\frac{U + ax}{ff - 2fgx \cos \phi + ggxx}$$

& complementum ad propositam $\frac{P}{Q}$ fit $= \frac{V}{S}$,

$$V = \frac{P - (U + ax)S}{ff - 2fgx \cos \phi + ggxx}; \text{ unde } P - (U + ax)S$$

ac propterea quoque $\frac{P}{S} - U - ax$ divisibile erit per

$ff - 2fgx \cos \phi + ggxx$. Evanesct ergo $\frac{P}{S} - U - ax$ si

ponatur $ff - 2fgx \cos \phi + ggxx = 0$, hoc est si ponatur

$$\text{vel } x = \frac{f}{g} \cos \phi + \frac{f}{g\sqrt{-1}} \sin \phi$$

$$\text{vel } x = \frac{f}{g} \cos \phi - \frac{f}{g\sqrt{-1}} \sin \phi.$$

414. Quoniam P & S sunt functiones integrae ipsius x , fiat in utroque seorsim utraque substitutio; & quia pro quavis potestate ipsius x , puta x^n binomium hoc $x^n = \frac{f^n}{g^n} \cos^n \phi \pm \frac{f^n}{g^n \sqrt{-1}} \sin^n \phi$ substitui debet. Ponamus primo ubique $\frac{f^n}{g^n} \cos^n \phi$ pro x^n , hocque facto abeat P in \mathfrak{P} ,

& S in \mathfrak{S} . Deinde ponatur ubique $\frac{f^n}{g^n} \sin^n \phi$ pro x^n , hocque facto abeat P in \mathfrak{p} & S in \mathfrak{s} ; ubi notandum est ante

has substitutiones utramque functionem P & S penitus debere evolvi, ita ut, si forte factoribus sint implicatae, ii per actualem multiplicationem tollantur. His valoribus \mathfrak{P} , \mathfrak{p} , \mathfrak{S} , \mathfrak{s} , inventis, manifestum erit, si ponatur:

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Etionem $\frac{dQ}{dx}$ abire in Ω ; sin autem statuatur $x^n = \frac{f^n}{g^n} \sin^n \phi$,
 cam abire in q ; atque manifestum est si ponatur

$$x = \frac{f}{g} \cos \phi \pm \frac{f}{g\sqrt{-1}} \sin \phi$$

functionem $\frac{dQ}{dx}$ abire in $\Omega \pm \frac{q}{\sqrt{-1}}$. Ex quo functio S

abibit in $\frac{\Omega \pm q : \sqrt{-1}}{\pm 2fg \sin \phi : \sqrt{-1}}$. Cum ergo sit $S = \frac{\mathcal{C} \pm \mathcal{S}}{\sqrt{-1}}$

eodem valore pro x posito, habebitur,

$$\Omega \pm \frac{q}{\sqrt{-1}} = \pm \frac{2fg\mathcal{C}}{\sqrt{-1}} \sin \phi - 2fg\mathcal{S} \sin \phi.$$

Erit ergo $\mathcal{S} = \frac{-\Omega}{2fg \sin \phi}$ & $\mathcal{C} = \frac{q}{2fg \sin \phi}$.

Hisque valoribus substitutis, fiet $a = \frac{2gg(pq + \mathcal{P}\Omega)}{\Omega^2 + q^2}$

$$\& \mathcal{R} = \frac{2fg(\mathcal{P}q - p\Omega) \sin \phi}{\Omega^2 + q^2} \frac{2fg(pq + \mathcal{P}\Omega) \cos \phi}{\Omega^2 + q^2}.$$

416. Hinc ergo idonea obtinetur ratio ex quovis facto-
 re secundae potestatis fractionem simplicem formandi, hicque
 cum ipse fractionis propositae denominator in computo reti-
 neatur, divisionem, qua valor litterae S definiri deberet, &
 quae saepe non parum est molesta, evitamus. Si igitur fra-
 ctionis $\frac{P}{Q}$ denominator Q factorem habeat talem $ff - 2fgx \cos \phi$
 $+ ggxx$, sequenti modo fractio simplex ex hoc factore
 oriunda, quam fingamus $= \frac{u + ax}{ff - 2fgx \cos \phi + ggxx}$, definie-
 tur. Ponatur $x = \frac{f}{g} \cos \phi$, & pro quavis ipsius x potestate

x^n scribatur $\frac{f^n}{g^n} \cos n\phi$; quo facto abeat P in \mathfrak{P} , & functio $\frac{dQ}{dx}$ in Ω . Deinde ibidem ponatur $x = \frac{f}{g} \sin \phi$, & pote-

stas eius quaevis $x^n = \frac{f^n}{g^n} \sin n\phi$; abeatque P in p , & $\frac{dQ}{dx}$ in q . Inventisque hoc modo valoribus litterarum \mathfrak{P} , Ω , p & q quantitates \mathfrak{U} & \mathfrak{a} ita definientur, ut sit

$$\mathfrak{U} = \frac{2fg(\mathfrak{P}q - p\Omega)\sin\phi}{\Omega^2 + q^2} - \frac{2fg(\mathfrak{P}\Omega + pq)\cos\phi}{\Omega^2 + q^2}$$

$$\mathfrak{a} = \frac{2gg(\mathfrak{P}\Omega + pq)}{\Omega^2 + q^2}.$$

Fractio ergo ex denominatoris Q factore $ff - 2fgx \cos \phi + ggxx$ oriunda erit:

$$\frac{2fg(\mathfrak{P}q - p\Omega)\sin\phi + 2g(\mathfrak{P}\Omega + pq)(gx - f\cos\phi)}{(\Omega^2 + q^2)(ff - 2fgx \cos \phi + ggxx)}$$

E X E M P L U M I.

Si proposita fuerit haec fractio $\frac{x^m}{a + bx^n}$, cuius denominator $a + bx^n$ factorem habeat hunc: $ff - 2fgx \cos \phi + ggxx$ invenire fractionem simplicem huic factori convenientem.

Quoniam hic est $P = x^m$ & $Q = a + bx^n$, erit

$$\frac{dQ}{dx} = nbx^{n-1}, \text{ unde fiet:}$$

$$\mathfrak{P} = \frac{f^m}{g^m} \cos m\phi \quad ; \quad p = \frac{f^m}{g^m} \sin m\phi$$

$$Q = \frac{nb f^{n-1}}{g^{n-1}} \cos(n-1)\phi \quad ; \quad q = \frac{nb f^{n-1}}{g^{n-1}} \sin(n-1)\phi.$$

$$\text{Ex his erit: } \Omega^2 + q^2 = \frac{n^2 b^2 f^{2(n-1)}}{g^{2(n-1)}} ;$$

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$\mathfrak{P} q$

$$\mathfrak{P}q - p\Omega = \frac{nb f^{m+n-1}}{g^{m+n-1}} \sin(n-m-1)\Phi;$$

$$\text{atque } \mathfrak{P}\Omega + p q = \frac{nb f^{m+n-1}}{g^{m+n-1}} \cos(n-m-1)\Phi.$$

Quamobrem erit fractio simplex quaesita:

$$\frac{2g^{n-m} [f \sin \Phi \cdot \sin(n-m-1)\Phi + g x \cos(n-m-1)\Phi - f \cos \Phi \cdot \cos(n-m-1)\Phi]}{nb f^{n-m-1} (\mathfrak{f}\mathfrak{f} - 2fgx \cos \Phi + ggxx)}$$

seu

$$\frac{2g^{n-m} [gx \cos(n-m-1)\Phi - f \cos(n-m)\Phi]}{nb f^{n-m-1} (\mathfrak{f}\mathfrak{f} - 2fgx \cos \Phi + ggxx)}$$

EXEMPLUM II.

Sit proposita haec fractio $\frac{1}{x^m(a+bx^n)}$, cuius denominator fractionem habeat $\mathfrak{f}\mathfrak{f} - 2fgx \cos \Phi + ggxx$, invenire fractionem simplicem inde oriundam.

Cum sit $P=1$, & $Q = ax^m + bx^{m+n}$,

erit $\frac{dQ}{dx} = m ax^{m-1} + (m+n)bx^{m+n-1}$, ideoque posito

$$x^n = \frac{f^n}{g^n} \cos n\Phi \text{ ob } P = x^0 \text{ \& } \mathfrak{P} = 1.$$

$$\Omega = \frac{maf^{m-1}}{g^{m-1}} \cos(m-1)\Phi + \frac{(m+n)bf^{m+n-1}}{g^{m+n-1}} \cos(m+n-1)\Phi$$

& $p=0$; atque

$$q = \frac{maf^{m-1}}{g^{m-1}} \sin(m-1)\Phi + \frac{(m+n)bf^{m+n-1}}{g^{m+n-1}} \sin(m+n-1)\Phi$$

Ergo

$$\Omega^2 + q^2 = \frac{m^2 a^2 f^{2(m-1)}}{g^{2(m-1)}} + \frac{2m(m+n)abf^{2m+n-2}}{g^{2m+n-2}} \cos n\Phi + \frac{(m+n)^2 b^2 f^{2(m+n-1)}}{g^{2(m+n-1)}}$$

Quodsi

Quodsi vero est $ff - 2fg \cos \phi + gg \cos^2 \phi$ divisor ipsius
 $a + bx^n$, erit $a + \frac{bf^n}{g^n} \cos n\phi = 0$ & $\frac{bf^n}{g^n} \sin n\phi = 0$,

unde: $aa = \frac{bbf^{2n}}{g^{2n}}$. Erit ergo:

$$\begin{aligned} \Omega + \Omega^2 &= \frac{(m+n)^2 bbf^{2(m+n-1)}}{g^{2(m+n-1)}} - \frac{m(2n+m) aaf^{2(m-1)}}{g^{2(m-1)}} \\ &= \frac{nn aaf^{2(m-1)}}{g^{2(m-1)}} = \frac{nn bbf^{2(m+n-1)}}{g^{2(m+n-1)}}, \end{aligned}$$

Deinde vero erit:

$$\begin{aligned} \mathfrak{P}q - p\Omega &= \frac{maf^{m-1}}{g^{m-1}} \sin(m-1)\phi + \frac{(m+n)bf^{m+n-1}}{g^{m+n-1}} \sin(m+n-1)\phi \\ &= \frac{bf^{m+n-1}}{g^{m+n-1}} [(m+n) \sin(m+n-1)\phi - m \cos n\phi \cdot \sin(m-1)\phi] \\ &= \frac{bf^{m+n-1}}{g^{m+n-1}} [n \cos n\phi \cdot \sin(m-1)\phi + (m+n) \sin n\phi \cos(m-1)\phi] \\ &= \frac{bf^{m+n-1}}{g^{m+n-1}} [(m+n) \cos(m+n-1)\phi - m \cos n\phi \cdot \cos(m-1)\phi] \end{aligned}$$

& $\mathfrak{P}\Omega + pq =$

Vel cum $ff - 2fg \cos \phi + gg \cos^2 \phi$ fit quoque divisor ipsius
 $ax^{m-1} + bx^{m+n-1}$, erit:

$$\begin{aligned} \frac{af^{m-1}}{g^{m-1}} \cos(m-1)\phi + \frac{bf^{m+n-1}}{g^{m+n-1}} \cos(m+n-1)\phi &= 0 \\ \& \frac{af^{m-1}}{g^{m-1}} \sin(m-1)\phi + \frac{bf^{m+n-1}}{g^{m+n-1}} \sin(m+n-1)\phi &= 0, \end{aligned}$$

unde erit:

$$\Omega = \frac{nbf^{m+n-1}}{g^{m+n-1}} \cos(m+n-1)\phi \quad \& \quad q = \frac{nbf^{m+n-1}}{g^{m+n-1}} \sin(m+n-1)\phi$$

seu

$$\Omega = \frac{-naf^{m-1}}{g^{m-1}} \cos(m-1)\phi \quad \& \quad q = \frac{-naf^{m-1}}{g^{m-1}} \sin(m-1)\phi.$$

Ex quibus resultabit fractio quaesita :

$$+ \frac{2g^m [f \cos m\phi - g \cos(m-1)\phi]}{n a f^{n-1} (ff - 2fg \cos \phi + gg \cos^2 \phi)}$$

Quae formula ex priori exemplo sequitur, si ponatur m negativum, unde non opus fuit hunc casum peculiarem constituisse.

EXEMPLUM III.

Si huius fractionis $\frac{x^m}{a + bx^n + cx^{2n}}$ denominator habuerit factorem $ff - 2fg \cos \phi + gg \cos^2 \phi$, fractionem simplicem investigare ex hoc factore erundam.

Si $ff - 2fg \cos \phi + gg \cos^2 \phi$ est factor denominatoris $a + bx^n + cx^{2n}$, erit ut supra ostendimus:

$$a + \frac{bf^n}{g^n} \cos n\phi + \frac{cf^{2n}}{g^{2n}} \cos 2n\phi = 0$$

$$\& \frac{bf^n}{g^n} \sin n\phi + \frac{cf^{2n}}{g^{2n}} \sin 2n\phi = 0.$$

Cum igitur sit $P = x^m$ & $Q = a + bx^n + cx^{2n}$,

erit $\frac{dQ}{dx} = nbx^{n-1} + 2ncx^{2n-1}$; unde efficitur:

$$p = \frac{f^m}{g^m} \cos m\phi \quad \& \quad q = \frac{f^m}{g^m} \sin m\phi$$

$$Q = \frac{nbf^{n-1}}{g^{n-1}} \cos(n-1)\phi + \frac{2ncf^{2n-1}}{g^{2n-1}} \cos(2n-1)\phi$$

$$q = \frac{nbf^{n-1}}{g^{n-1}} \sin(n-1)\phi + \frac{2ncf^{2n-1}}{g^{2n-1}} \sin(2n-1)\phi$$

Quamobrem habebimus:

$$Q^2 + q^2 = \frac{n^2 f^{2(n-1)}}{g^{2(n-1)}} \left(bb + \frac{4bcf^n}{g^n} \cos n\phi + \frac{4ccf^{2n}}{g^{2n}} \right)$$

At ex duabus prioribus aequationibus est:

$$\frac{f^{2n}}{g^{2n}} \left(bb + \frac{2bcf^n}{g^n} \cos n\phi + \frac{ccf^{2n}}{g^{2n}} \right) = a^2; \quad \text{ideo}$$

ideoque

$$\frac{4bcf^n}{g^n} \cos n\phi = \frac{2g^{2n}aa}{f^{2n}} - 2bb - \frac{2ccf^{2n}}{g^{2n}}$$

quo valore ibi substituto erit:

$$\Omega^2 + \varrho^2 = \frac{n^2 f^{2n-2}}{g^{2n-2}} \left(\frac{2aa g^{2n}}{f^{2n}} - bb + \frac{2ccf^{2n}}{g^{2n}} \right)$$

seu

$$\Omega^2 + \varrho^2 = \frac{n^2 (2aa g^{4n} - bb f^{2n} g^{2n} + 2cc f^{4n})}{ff g^{4n-2}}$$

Deinde erit: $\mathfrak{P}\varrho - p\Omega =$

$$\frac{nb f^{m+n-1}}{g^{m+n-1}} \sin(n-m-1)\phi + \frac{2nc f^{m+2n-1}}{g^{m+2n-1}} \sin(2n-m-1)\phi$$

$\mathfrak{P}\Omega + p\varrho =$

$$\frac{nb f^{m+n-1}}{g^{m+n-1}} \cos(n-m-1)\phi + \frac{2nc f^{m+2n-1}}{g^{m+2n-1}} \cos(2n-m-1)\phi.$$

Quibus valoribus inventis erit fractio simplex quaesita:

$$\frac{2fg(\mathfrak{P}\varrho - p\Omega) \sin \phi + 2g(\mathfrak{P}\Omega + p\varrho)(g^n - f \cos \phi)}{(\Omega^2 + \varrho^2)(ff - 2fg \cos \phi + gg \cos^2 \phi)}$$

417. Hae autem fractiones facilius exprimentur, si ipsos denominatorum factores determinemus. Sit igitur denominator fractionis propositae: $a + bx^n$ cuius factor trinomialis si ponatur: $ff - 2fg \cos \phi + gg \cos^2 \phi$ erit uti in Introductione ostendimus:

$$a + \frac{bf^n}{g^n} \cos n\phi = 0 \quad \& \quad \frac{bf^n}{g^n} \sin n\phi = 0,$$

cum igitur sit $\sin n\phi = 0$, erit vel $n\phi = (2k-1)\pi$, vel $n\phi = 2k\pi$, priori casu erit $\cos n\phi = -1$, posteriori $\cos n\phi = +1$. Si ergo a & b sint quantitates affirmativae,

prior casus solus locum habebit, quo fit $a = \frac{bf^n}{g^n}$; ac

propterea: $f = a^{\frac{1}{n}}$ & $g = b^{\frac{1}{n}}$

reti-

retineamus autem loco harum quantitatum irrationalium litteras f & g , seu ponamus potius $a = f^n$ & $b = g^n$, ita ut factores investigari debeant huius functionis $f^n + g^n x^n$.

Cum igitur sit $\phi = \frac{(2k-1)\pi}{n}$, ubi k numerum quemcunque affirmativum integrum designare potest; at vero maiores numeri pro k non sunt sumendi, quam qui reddant $\frac{2k-1}{n}$ unitate minorem; hinc functionis propositae $f^n + g^n x^n$ factores erunt sequentes:

$$ff - 2fgx \cos \frac{\pi}{n} + ggxx$$

$$ff - 2fgx \cos \frac{3\pi}{n} + ggxx$$

$$ff - 2fgx \cos \frac{5\pi}{n} + ggxx$$

&c.

ubi notandum est si n sit numerus impar, unum factorem haberi binomium hunc: $f + gx$; sin autem n sit numerus par nullus factor aderit binomius.

EXEMPLUM I.

Resolvere hanc fractionem $\frac{x^m}{f^n + g^n x^n}$ in suas fractiones simplices.

Cum denominatoris unusquisque factor trinomialis contineatur in hac forma: $ff - 2fgx \cos \frac{(2k-1)\pi}{n} + ggxx$, erit in §. praecedente Exempl. I. $a = f^n$, $b = g^n$, & $\phi = \frac{(2k-1)\pi}{n}$, unde erit:

$$\sin(n-m-1)\phi = \sin(m+1)\phi = \sin \frac{(m+1)(2k-1)\pi}{n} \quad \&c$$

cof

$Ax^{m-n} + Bx^{m-2n} + Cx^{m-3n} + Dx^{m-4n} + \&c.$
 quamdiu exponentes manent affirmativi, eritque:

$$Ag^n = 1; \text{ Ergo } A = \frac{1}{g^n}$$

$$Af^n + Bg^n = 0 \dots B = -\frac{f^n}{g^{2n}}$$

$$Bf^n + Cg^n = 0 \dots C = +\frac{f^{2n}}{g^{3n}}$$

$$Cf^n + Dg^n = 0 \dots D = -\frac{f^{3n}}{g^{4n}}$$

&c.

&c.

EXEMPLUM II.

Resolvere hanc fractionem $\frac{1}{x^m (f^n + g^n x^n)}$ in suas
 fractiones simplices.

Quod ad factores ipsius $f^n + g^n x^n$ attinet, ex iis oriuntur
 eadem fractiones; quas Exemplo praecedente eruimus, dum-
 modo ibi sumatur m negative: superest igitur tantum, ut
 fractiones simplices ex denominatoris altero factore x^m defi-
 niamus, quod hoc modo commodissime fit: statuatur fractio

$$\text{proposita} = \frac{U}{x^m} + \frac{Vx^{n-m}}{f^n + g^n x^n}, \text{ eritque}$$

$$Uf^n = 1; \text{ Ergo } U = \frac{1}{f^n}$$

$$Ug^n + V = 0 \dots V = -\frac{g^n}{f^n}.$$

Si $n-m$ adhuc fuerit numerus negativus, simili modo erit
 operandum, ita ut, si m fuerit numerus quantumvis magnus,
 resultent huiusmodi fractiones simplices

$$\frac{U}{x^m} + \frac{B}{x^{m-n}} + \frac{C}{x^{m-2n}} + \frac{D}{x^{m-3n}} + \&c.$$

cuius

eius seriei tot termini sunt sumendi, quot habentur ipsius
 * exponentes affirmativi in denominatore. Eritque

$$\mathcal{A}f^n = 1 \quad ; \quad \text{Ergo} \quad \mathcal{A} = \frac{1}{f^n}$$

$$\mathcal{A}g^n + \mathcal{B}f^n = 0 \quad . \quad . \quad \mathcal{B} = -\frac{g^n}{f^{2n}}$$

$$\mathcal{B}g^n + \mathcal{C}f^n = 0 \quad . \quad . \quad \mathcal{C} = +\frac{g^{2n}}{f^{3n}}$$

$$\mathcal{C}g^n + \mathcal{D}f^n = 0 \quad . \quad . \quad \mathcal{D} = -\frac{g^{3n}}{f^{4n}}$$

&c. &c.

Fractio ergo proposita omnino in has fractiones simplices
 resolvetur :

$$\frac{1}{f^n x^m} - \frac{g^n}{f^{2n} x^{m-n}} + \frac{g^{2n}}{f^{3n} x^{m-2n}} - \frac{g^{3n}}{f^{4n} x^{m-3n}} + \&c.$$

$$\frac{-2fg^m \sin \frac{\pi}{n} \cdot \sin \frac{(m-1)\pi}{n} - 2g^m \cos \frac{(m-1)\pi}{n} \left(g^n - f \cos \frac{\pi}{n} \right)}{nf^{n+m-1} \left(ff - 2fgx \cos \frac{\pi}{n} + ggxx \right)}$$

$$\frac{-2fg^m \sin \frac{3\pi}{n} \cdot \sin \frac{3(m-1)\pi}{n} - 2g^m \cos \frac{3(m-1)\pi}{n} \left(g^n - f \cos \frac{3\pi}{n} \right)}{nf^{n+m-1} \left(ff - 2fgx \cos \frac{3\pi}{n} + ggxx \right)}$$

$$\frac{-2fg^m \sin \frac{5\pi}{n} \cdot \sin \frac{5(m-1)\pi}{n} - 2g^m \cos \frac{5(m-1)\pi}{n} \left(g^n - f \cos \frac{5\pi}{n} \right)}{nf^{n+m-1} \left(ff - 2fgx \cos \frac{5\pi}{n} + ggxx \right)}$$

&c.

Quibus formulis si n fuerit numerus impar, ob $f + \frac{g^n}{f}$ fa-
 cto.

Tttt

Storem denominatoris, insuper adici debet:

$$\frac{\pm g^m}{nf^{n+m-1}(f+gx)}$$

ubi signorum ambiguum \pm superius valet, si m fuerit numerus par, inferius vero si m impar.

418. Consideremus nunc quoque formulam $a+bx^n$, si b sit numerus negativus, sitque proposita haec functio:

$$f^n - g^n x^n$$

cuius primo semper erit factor $f - gx$; atque si n sit numerus par, quoque $f + gx$ eius erit factor. Reliqui vero erunt trinomiales, quorum forma generalis si ponatur

$$ff - 2fgx \cos \phi + ggxx$$

erit $f^n - f^n \cos n\phi = 0$ & $f^n \sin n\phi = 0$ sive $\sin n\phi = 0$ & $\cos n\phi = 1$. Quibus ut satisfiat, oportet esse $n\phi = 2k\pi$ existente k numero quocunque integro, atque propterea erit $\phi = \frac{2k\pi}{n}$. Factor ergo generalis erit:

$$ff - 2fgx \cos \frac{2k\pi}{n} + ggxx$$

sumendo ergo pro $2k$ omnes numeros pares exponente n minores, prodibunt factores trinomiales omnes:

$$ff - 2fgx \cos \frac{2\pi}{n} + ggxx$$

$$ff - 2fgx \cos \frac{4\pi}{n} + ggxx$$

$$ff - 2fgx \cos \frac{6\pi}{n} + ggxx$$

&c.

EXEMPLUM I.

Resolvère hanc fractionem $\frac{x^m}{f^n - g^n x^n}$ in suas

fractiones simplices.

Quoniam denominatoris factor est $f - gx$, inde orietur fra-

fractio huiusmodi $\frac{\mathcal{U}}{f - g^x}$, ad cuius numeratorem invenien-
dum, ponatur $x^m = P$ & $f^n - g^n x^n = Q$, erit $\frac{dQ}{dx} = -ng^n x^{n-1}$,
fietque $\mathcal{U} = \frac{-g^m x^m}{-ng^n x^{n-1}} = \frac{g^m x^m}{ng^{n-1} x^{n-1}}$, posito $x = \frac{f}{g}$. Ergo
erit $\mathcal{U} = \frac{1}{nf^{n-m-1} g^m}$, hincque fractio simplex ex factore
 $f - g^x$ orta erit:

$$\frac{1}{nf^{n-m-1} g^m (f - g^x)}$$

Si n sit numerus par, quia tum denominatoris factor quoque
est $f + g^x$, ponatur fractio simplex inde oriunda.

$$= \frac{\mathcal{U}}{f + g^x}, \text{ erit } \mathcal{U} = \frac{-g^m x^m}{ng^n x^{n-1}} = \frac{-x^m}{ng^{n-1} x^{n-1}},$$

posito $x = \frac{f}{g}$. Fiet ergo ob $n - 1$ numerum imparem

$g^{n-1} x^{n-1} = -f^{n-1}$: at erit $x^m = \frac{+f^m}{g^m}$, ubi signum supe-
rius valet, si m fuerit numerus par, inferius si m sit nu-
merus impar. Quare cum sit $\mathcal{U} = \frac{\mp 1}{nf^{n-m-1} g^m}$, erit fractio
simplex ex factore $f + g^x$ oriunda haec:

$$\frac{\mp 1}{nf^{n-m-1} g^m (f + g^x)}$$

Deinde cum factorum trinomialium forma generalis sit:

$$ff - 2fg^x \cos \frac{2k\pi}{n} + gg^x x,$$

si comparatio cum Exemplo I. §. 416. instituat, erit $a = f^n$,

$b = -g^n$ & $\phi = \frac{2k\pi}{n}$; unde $\sin n\phi = 0$ & $\cos n\phi = 1$;

$$\text{atque } \sin(n-m-1)\phi = -\sin(m+1)\phi = -\sin \frac{2k(m+1)\pi}{n},$$

$$\& \text{ cof}(n-m-1)\phi = \text{cof}(m+1)\phi = \text{cof} \frac{2k(m+1)\pi}{n}.$$

Ex quibus erit fractio simplex hinc oriunda :

$$\frac{2f \sin \frac{2k\pi}{n} \cdot \sin \frac{2k(m+1)\pi}{n} - 2 \text{cof} \frac{2k(m+1)\pi}{n} (g\kappa - f \text{cof} \frac{2k\pi}{n})}{n f^{n-m-1} g^m \left(ff - 2fg\kappa \text{cof} \frac{2k\pi}{n} + gg\kappa\kappa \right)}$$

Hancobrem fractiones simplices quaesitae erunt :

$$\frac{\frac{1}{n f^{n-m-1} g^m (f - g\kappa)}}{+ 2f \sin \frac{2\pi}{n} \cdot \sin \frac{2(m+1)\pi}{n} - 2 \text{cof} \frac{2(m+1)\pi}{n} (g\kappa - f \text{cof} \frac{2\pi}{n})}$$

$$n f^{n-m-1} g^m \left(ff - 2fg\kappa \text{cof} \frac{2\pi}{n} + gg\kappa\kappa \right)$$

$$+ 2f \sin \frac{4\pi}{n} \cdot \sin \frac{4(m+1)\pi}{n} - 2 \text{cof} \frac{4(m+1)\pi}{n} (g\kappa - f \text{cof} \frac{4\pi}{n})$$

$$n f^{n-m-1} g^m \left(ff - 2fg\kappa \text{cof} \frac{4\pi}{n} + gg\kappa\kappa \right)$$

$$+ 2f \sin \frac{6\pi}{n} \cdot \sin \frac{6(m+1)\pi}{n} - 2 \text{cof} \frac{6(m+1)\pi}{n} (g\kappa - f \text{cof} \frac{6\pi}{n})$$

$$n f^{n-m-1} g^m \left(ff - 2fg\kappa \text{cof} \frac{6\pi}{n} + gg\kappa\kappa \right)$$

&c.

quibus si n fuerit numerus par, insuper addi debet haec fractio :

$$\frac{\frac{1}{n f^{n-m-1} g^m (f + g\kappa)}}{}$$

cu-

cuius
par,
mino
quam

F
tur,
nega
picie

quae
fiant

cuius signum superius — est sumendum, si m fuerit numerus par, inferius si impar. Praeterea vero si m sit numerus non minor quam n , adiciendae sunt partes integrae:

$$A x^{m-n} + B x^{m-2n} + C x^{m-3n} + D x^{m-4n} + \&c.$$

quamdiu exponentes non fuerint negativi, eritque:

$$-A g^n = 1 \quad \text{feu} \quad A = -\frac{1}{g^n}$$

$$A f^n - B g^n = 0 \quad ; \quad ; \quad B = -\frac{f^n}{g^{2n}}$$

$$B f^n - C g^n = 0 \quad . \quad . \quad C = -\frac{f^{2n}}{g^{3n}}$$

$$C f^n - D g^n = 0 \quad . \quad . \quad D = -\frac{f^{3n}}{g^{4n}}$$

&c. &c.

E X E M P L U M II.

Resolvere hanc fractionem $\frac{1}{x^m (f^n - g^n x^n)}$ in suas

fractiones simplices

Fractiones quae ex denominatoris factore $f^n - g^n x^n$ oriuntur, eadem erunt quae ante, dummodo in illis formulis m negative accipiatur. Quare ad alterum factorem x^m est respiciendum, ex quo si ponamus has fractiones resultare:

$$\frac{A}{x^m} + \frac{B}{x^{m-n}} + \frac{C}{x^{m-2n}} + \frac{D}{x^{m-3n}} + \&c.$$

quae series eousque est continuanda, donec exponentes ipsius x fiant negativi. Erit vero

$$A f^n = 1 \quad ; \quad \text{Ergo} \quad A = \frac{1}{f^n}$$

$$B f^n - A g^n = 0 \quad . \quad . \quad B = \frac{g^n}{f^{2n}}$$

$$C f^n - B g^n = 0 \quad . \quad . \quad C = \frac{g^{2n}}{f^{3n}}$$

$$D f^n - C g^n = 0 \quad . \quad . \quad D = \frac{g^{3n}}{f^{4n}}$$

&c.

&c.

Fra-

Fractio ergo proposita resolvetur in has fractiones simplices:

$$\frac{1}{f^n x^m} + \frac{g^n}{f^{2n} x^{m-n}} + \frac{g^{2n}}{f^{3n} x^{m-2n}} + \frac{g^{3n}}{f^{4n} x^{m-3n}} + \&c.$$

$$+ \frac{g^m}{n f^{n+m-1} (f - g x)}$$

$$- 2 f g^m \sin \frac{2\pi}{n} \sin \frac{2(m-1)\pi}{n} - 2 g^m \cos \frac{2(m-1)\pi}{n} \left(g x - f \cos \frac{2\pi}{n} \right)$$

$$n f^{n+m-1} \left(f f - 2 f g x \cos \frac{2\pi}{n} + g g x x \right)$$

$$- 2 f g^m \sin \frac{4\pi}{n} \sin \frac{4(m-1)\pi}{n} - 2 g^m \cos \frac{4(m-1)\pi}{n} \left(g x - f \cos \frac{4\pi}{n} \right)$$

$$n f^{n+m-1} \left(f f - 2 f g x \cos \frac{4\pi}{n} + g g x x \right)$$

$$- 2 f g^m \sin \frac{6\pi}{n} \sin \frac{6(m-1)\pi}{n} - 2 g^m \cos \frac{6(m-1)\pi}{n} \left(g x - f \cos \frac{6\pi}{n} \right)$$

$$n f^{n+m-1} \left(f f - 2 f g x \cos \frac{6\pi}{n} + g g x x \right)$$

&c.

quibus si n fuerit numerus par, insuper addi debet haec fractio:

$$\mp \frac{g^m}{n f^{n+m-1} (f + g x)}$$

quae autem praetermittitur, si n fuerit numerus impar. Si-
gnorum ambiguum vero superius $-$ valet, si m sit nume-
rus par, inferius vero $+$, si m sit numerus impar.

419. Hoc ergo modo omnes fractiones, quarum deno-
minator ex duobus constat membris huiusmodi $a + b x^n$, in
fractiones simplices resolvuntur. At si denominator constet tri-
bus huiusmodi membris $a + b x^n + c x^{2n}$, tum primum viden-
dum est, utrum is in duos factores reales prioris formae re-
solvatur. Hoc enim si eveniat, resolutio in fractiones sim-
pli-

plices modo ante exposito institui poterit. Si enim proponatur huiusmodi fractio

$$\frac{x^m}{(f^n + g^n x^n)(f^n + b^n x^n)}$$

ea primum in duas fractiones transformabitur huiusmodi:

$$\frac{\alpha x^m}{f^n + g^n x^n} + \frac{\beta x^m}{f^n + b^n x^n}$$

eritque $\alpha f^n + \beta f^n = 1$ & $\alpha b^n + \beta g^n = 0$, unde fit

$$\alpha = \frac{1}{f^n} - \beta = -\frac{\beta g^n}{b^n}; \text{ ideoque habebitur}$$

$$\beta = \frac{b^n}{f^n(b^n - g^n)} \quad \& \quad \alpha = \frac{g^n}{f^n(g^n - b^n)}.$$

Si exponens m fuerit maior quam n , transmutatio in sequentes fractiones erit commodior:

$$\frac{\alpha x^{m-n}}{f^n + g^n x^n} + \frac{\beta x^{m-n}}{f^n + b^n x^n}$$

qua fit $\alpha + \beta = 0$ & $\alpha b^n + \beta g^n = 1$, ideoque $\alpha = \frac{1}{b^n - g^n}$

& $\beta = \frac{1}{g^n - b^n}$. Utra autem transformatio adhibeatur, utra-

que fractio hoc modo oriunda methodo ante exposita resolvatur in suas fractiones simplices, quae iunctim sumtae fractioni propositae erunt aequales.

420. Simili modo methodus haec tradita sufficet, si denominator ex pluribus membris constet huiusmodi formae $f^n \pm g^n x^n$ resolvi queat. Ponamus enim occurrere hanc fractionem in suas fractiones simplices resolvendam:

$$\frac{x^n}{(a-x^n)(b-x^n)(c-x^n)(d-x^n)}$$

Haec primum resolvetur in has:

$$\frac{A x^n}{a-x^n} + \frac{B x^n}{b-x^n} + \frac{C x^n}{c-x^n} + \frac{D x^n}{d-x^n}$$

quarum numeratores sequenti modo determinabuntur, ut fit

$$A = \frac{1}{(b-a)(c-a)(d-a)}$$

$$B = \frac{1}{(a-b)(c-b)(d-b)}$$

$$C = \frac{1}{(a-c)(b-c)(d-c)} \quad \&c.$$

Hac ergo praeparatione facta, singulae istae fractiones methodo ante exposita in suas fractiones simplices resolvuntur; quae cunctae in unam summam erunt colligendae.

421. Quodsi vero huiusmodi denominator $a + b x^n + c x^{2n} + d x^{3n} + \&c.$ non omnes factores formae $f^n + g^n x^n$ habeant reales, bini imaginarii erunt coniungendi. Ponamus ergo huiusmodi binorum factorum productum esse:

$$f^{2n} - 2 f^n g^n x^n \cos \omega + g^{2n} x^{2n}$$

& cum haec expressio nullos habeat factores simplices reales, ponamus factores trinomiales in hac forma generali contineri:

$$ff - 2fgx \cos \phi + ggxx$$

quorum numerus erit $= n$. Posito ergo $x^n = \frac{f^n}{g^n} \cos n\phi$;

oriatur haec aequatio:

$$1 - 2 \cos \omega \cdot \cos n\phi + \cos 2n\phi = 0.$$

Deinde posito $x^n = \frac{f^n}{g^n} \sin n\phi$; erit quoque:

$$-2 \cos \omega \cdot \sin n\phi + \sin 2n\phi = 0$$

quae divisa per $\sin n\phi$ dat $\cos n\phi = \cos \omega$, sicque simul prio-

ri

ri aequationi satisfit. Erit ergo $n\phi = 2k\pi \pm \omega$ denotante k numerum quemvis integrum, ideoque erit $\phi = \frac{2k\pi \pm \omega}{n}$, & factores omnes continebuntur in hac forma:

$$ff - 2fgx \cos \frac{2k\pi \pm \omega}{n} + ggxx \text{ unde sequentes habebuntur fa-}$$

$$\text{ctores: } ff - 2fgx \cos \frac{\omega}{n} + ggxx$$

$$ff - 2fgx \cos \frac{2\pi - \omega}{n} + ggxx$$

$$ff - 2fgx \cos \frac{2\pi + \omega}{n} + ggxx$$

$$ff - 2fgx \cos \frac{4\pi - \omega}{n} + ggxx$$

$$ff - 2fgx \cos \frac{4\pi + \omega}{n} + ggxx \text{ \&c.}$$

quorum tot sunt sumendi, donec eorum numerus fiat $= n$.

422. Si igitur proponatur ista fractio in suas fractiones simplices resolvenda:

$$\frac{x^{m-1}}{f^{2n} - 2f^n g^n x^n \cos \omega + g^{2n} x^{2n}}$$

quoniam denominatoris factor trinomialis quicumque contine-

tur in hac forma: $ff - 2fgx \cos \phi + ggxx$

existente $\phi = \frac{2k\pi \pm \omega}{n}$, consideretur ista fractio:

$$\frac{x^m}{f^{2n} x - 2f^n g^n x^{n+1} \cos \omega + g^{2n} x^{2n+1}}$$

illi aequalis, ac ponatur numerator $x^m = P$ ac denominator $f^{2n} x - 2f^n g^n x^{n+1} \cos \omega + g^{2n} x^{2n+1} = Q$: erit

VVVV

dQ

$$\frac{dQ}{dx} = f^{2n} - 2(n+1)f^n g^n x^n \cos \omega + (2n+1)g^{2n} x^{2n}.$$

Hinc ponendo $x^n = \frac{f^n}{g^n} \cos n\phi$; erit: $\mathfrak{P} = \frac{f^m}{g^m} \cos m\phi$

feu $\mathfrak{P} = \frac{f^m}{g^m} \cos \frac{m(2k\pi \pm \omega)}{n}$ &

$\Omega = f^{2n} [1 - 2(n+1) \cos \omega \cos n\phi + (2n+1) \cos 2n\phi]$. Cum autem sit $\cos n\phi = \cos \omega$, erit $\cos 2n\phi = 2 \cos^2 \omega - 1$; ideoque $\Omega = f^{2n} (-2n + 2n \cos \omega^2) = -2n f^{2n} \sin \omega^2$.

Deinde posito $x^n = \frac{f^n}{g^n} \sin n\phi$, fiet:

$p = \frac{f^m}{g^m} \sin m\phi = \frac{f^m}{g^m} \sin \frac{m(2k\pi \pm \omega)}{n}$ &

$q = -f^{2n} [2(n+1) \cos \omega \sin n\phi - (2n+1) \sin 2n\phi]$
ob $\sin 2n\phi = 2 \sin n\phi \cos n\phi = 2 \cos \omega \sin n\phi$; erit
 $q = 2n f^{2n} \cos \omega \sin n\phi$. Cum autem sit $n\phi = 2k\pi \pm \omega$,
erit $\sin n\phi = \pm \sin \omega$ & $q = \pm 2n f^{2n} \sin \omega \cos \omega$.
His inventis erit: $\Omega^2 + q^2 = 4n^2 f^{4n} \sin \omega^2$

$\mathfrak{P}q - p\Omega = \frac{2n f^{m+2n}}{g^m} (\pm \cos m\phi \sin \omega \cos \omega \pm \sin m\phi \sin \omega^2)$

five $\mathfrak{P}q - p\Omega = \pm \frac{2n f^{m+2n}}{g^m} \sin \omega \cos(m\phi \mp \omega)$ feu

$\mathfrak{P}q - p\Omega = \pm \frac{2n f^{m+2n}}{g^m} \sin \omega \cos \frac{2km\pi \pm (m-n)\omega}{n}$

$\mathfrak{P}\Omega + pq = \frac{2n f^{m+2n}}{g^m} (-\cos m\phi \sin \omega^2 \pm \sin m\phi \sin \omega \cos \omega)$

$\mathfrak{P}\Omega + pq = \pm \frac{2n f^{m+2n}}{g^m} \sin \omega \sin(m\phi \mp \omega)$ feu

$\mathfrak{P}\Omega + pq = \pm \frac{2n f^{m+2n}}{g^m} \sin \omega \sin \frac{2km\pi \pm (m-n)\omega}{n}$

Hinc

Hinc ex denominatoris factore: $ff - 2fg \times \cos \frac{2k\pi \pm \omega}{n} + gg \times \times$

nascitur ista fractio simplex:

$$\frac{+ f \sin \frac{2k\pi \pm \omega}{n} \cos \frac{2km\pi \pm (m-n)\omega}{n} \pm \sin \frac{2km\pi \pm (m-n)\omega}{n} (g \times - f \cos \frac{2k\pi \pm \omega}{n})}{n f^{2n-m} g^{m-1} \sin \omega (ff - 2fg \times \cos \frac{2k\pi \pm \omega}{n} + gg \times \times)}$$

seu

$$\frac{\pm g \times \sin \frac{2km\pi \pm (m-n)\omega}{n} \mp f \sin \frac{2k(m-1)\pi \pm (m-n-1)\omega}{n}}{n f^{2n-m} g^{m-1} \sin \omega (ff - 2fg \times \cos \frac{2k\pi \pm \omega}{n} + gg \times \times)}$$

E X E M P L U M.

Resolvete hanc fractionem $\frac{f^{2n} - 2f^n g^n x^n \cos \omega + g^{2n} x^{2n}}{x^{m-1}}$

in suas fractiones simplices.

Istae fractiones simplices quaelitae ergo erunt:

$$\frac{+ f \sin \frac{\omega}{n} \cos \frac{(m-n)\omega}{n} \pm \sin \frac{(m-n)\omega}{n} (g \times - f \cos \frac{\omega}{n})}{n f^{2n-m} g^{m-1} \sin \omega (ff - 2fg \times \cos \frac{\omega}{n} + gg \times \times)}$$

$$\frac{- f \sin \frac{2\pi - \omega}{n} \cos \frac{2m\pi - (m-n)\omega}{n} - \sin \frac{2m\pi - (m-n)\omega}{n} (g \times - f \cos \frac{2\pi - \omega}{n})}{n f^{2n-m} g^{m-1} \sin \omega (ff - 2fg \times \cos \frac{2\pi - \omega}{n} + gg \times \times)}$$

$$\frac{\mp f \sin \frac{2\pi + \omega}{n} \cos \frac{2m\pi + (m-n)\omega}{n} \pm \sin \frac{2m\pi + (m-n)\omega}{n} (g \times - f \cos \frac{2\pi + \omega}{n})}{n f^{2n-m} g^{m-1} \sin \omega (ff - 2fg \times \cos \frac{2\pi + \omega}{n} + gg \times \times)}$$

$$n f^{2n-m} g^{m-1} \sin \omega (ff - 2fg \times \cos \frac{2\pi + \omega}{n} + gg \times \times)$$

$$\begin{aligned}
 & -f \sin \frac{4\pi - \omega}{n} \operatorname{cof} \frac{4m\pi - (m-n)\omega}{n} - \sin \frac{4m\pi - (m-n)\omega}{n} \left(g\kappa - f \operatorname{cof} \frac{4\pi - \omega}{n} \right) \\
 & \quad \frac{n f^{2n-m} g^{m-1} \sin \omega \left(ff - 2fg\kappa \operatorname{cof} \frac{4\pi - \omega}{n} + gg\kappa\kappa \right)}{n f^{2n-m} g^{m-1} \sin \omega \left(ff - 2fg\kappa \operatorname{cof} \frac{4\pi + \omega}{n} + gg\kappa\kappa \right)} \\
 & + f \sin \frac{4\pi + \omega}{n} \operatorname{cof} \frac{4m\pi + (m-n)\omega}{n} + \sin \frac{4m\pi + (m-n)\omega}{n} \left(g\kappa - f \operatorname{cof} \frac{4\pi + \omega}{n} \right) \\
 & \quad \frac{n f^{2n-m} g^{m-1} \sin \omega \left(ff - 2fg\kappa \operatorname{cof} \frac{4\pi + \omega}{n} + gg\kappa\kappa \right)}{n f^{2n-m} g^{m-1} \sin \omega \left(ff - 2fg\kappa \operatorname{cof} \frac{4\pi + \omega}{n} + gg\kappa\kappa \right)}
 \end{aligned}$$

&c.

ficque eoufque erit progrediendum, quoad harum fractionum numerus fuerit n . Si m fuerit numerus vel maior quam $2n-1$ vel numerus negativus, priori casu partes integrae, posteriori vero fractiones infuper sunt adiciendae, quae modo ante expofito facile inveniuntur.

FINIS.