INSTITUTIONUM
CALCULI DIFFERENTIALIS
PARS POSTERIOR
CONTINENS
USUM HUIUS CALCULI IN ANALYSI FINITORUM, NEC NON IN DOCTRINA SERIERUM.

CAPUT I.

DE TRANSFORMATIONE SERIERUM

I.

Cum nobis propositum sit usum Calculi differentialis tam in universa Analysi, quam in doctrina de seriebus ostendere; nonnulla subsidia ex Algebra comuni, quae vulgo tractari non solent, hic erunt repetenda. Quae quamvis maximam partem iam in introductione sumus complexi, tamen quaedam ibi sunt praetermissa, vel studio quod expediat ea tum demum explicari, quando necessitas id exigat, vel quia cuncta, quibus opus sit futurum, praevideri non poterant. Huc pertinet transformatio serierum, cui hoc Caput destinavimus, qua quaevis series in innumerabiles alias series transmutatur, quae omnes eandem habeant summam communem, ita ut, si seriei propositae summa sit cognita, reliquae series omnes simul summari queant. Hoc autem capite praemisso, eo uberius doctrinam serierum per calculum differentialem & integralem amplificare poterimus.

2. Considerabimus autem potissimum eiusmodi series, quarum singuli termini per potestates successivas quantitatis cuiusdam indeterminatae sunt multiplicati: quoniam hae latius patent, maioremque utilitatem afferent.

Sit igitur propoſita ſequens ſeries generalis, cuius ſummam, ſive ſit cognita ſive ſecus, ponamus $= S$, ſitque

$$S = ax + bx^2 + cx^3 + dx^4 + ex^5 + \&c.$$

Ponatur iam $x = \frac{y}{1+y}$, & cum ſit per ſeries infinitas

$$\begin{aligned} x &= y - y^2 + y^3 - y^4 + y^5 - y^6 + \&c. \\ x^2 &= y^2 - 2y^3 + 3y^4 - 4y^5 + 5y^6 - 6y^7 + \&c. \\ x^3 &= y^3 - 3y^4 + 6y^5 - 10y^6 + 15y^7 - 21y^8 + \&c. \\ x^4 &= y^4 - 4y^5 + 10y^6 - 20y^7 + 35y^8 - 56y^9 + \&c. \\ &\&c. \end{aligned}$$

hi valores ſubſtituti, ſerieque ſecundum poteſtates ipſius y diſpoſita, dabunt

$$\begin{aligned} S = ay &- ay^2 + ay^3 - ay^4 + ay^5 \&c. \\ &+ b \quad - 2b \quad + 3b \quad - 4b \\ &\qquad\quad + c \quad - 3c \quad + 6c \\ &\qquad\qquad\quad + d \quad - 4d \\ &\qquad\qquad\qquad\quad + e \end{aligned}$$

3. Quoniam poſuimus $x = \frac{y}{1+y}$; erit $y = \frac{x}{1-x}$, quo valore loco y ſubſtituto, ſeries propoſita

$$S = ax + bx^2 + cx^3 + dx^4 + ex^5 + \&c.$$

tranſmutabitur in hanc:

$$S = a.\frac{x}{1-x} + (b-a)\frac{x^2}{(1-x)^2} + (c-2b+a)\frac{x^3}{(1-x)^3} \&c.$$

in qua coefficiens ſecundi termini $b-a$ eſt differentia prima ipſius a ex ſerie a, b, c, d, e, &c. quam ſupra per Δa expoſuimus; coefficiens tertii termini $c-2b+a$ eſt differentia ſecunda $\Delta^2 a$; coefficiens quarti eſt differentia tertia $\Delta^3 a$, &c. Hinc differentiis ipſius a continuis, quae formantur ex ſerie a, b, c, d, e, &c. adhibendis propoſita ſeries tranſmutabitur in hanc

$$S = \frac{x}{1-x}a + \frac{x^2}{(1-x)^2}\Delta a + \frac{x^3}{(1-x)^3}\Delta^2 a + \frac{x^4}{(1-x)^4}\Delta^3 a + \&c.$$

cuius ergo ſeriei ſumma habebitur, ſi propoſitae ſumma fuerit cognita.

4.

4. Si igitur series a, b, c, d, &c. ita fuerit comparata, ut tandem differentias habeat constantes, quod evenit, si eius terminus generalis fuerit functio rationalis integra, tum series posterior $\frac{x}{1-x}a + \frac{x^2}{(1-x)^2}\Delta a +$ &c. tandem habebit terminos evanescentes, sicque eius summa per expressionem finitam exhiberi poterit. Ita si seriei a, b, c, d, &c. differentiae primae iam fuerint constantes, tum seriei huius: $ax + bx^2 + cx^3 + dx^4 +$ &c. summa erit $= \frac{x}{1-x}a + \frac{x^2}{(1-x)^2}\Delta a$. At si illius seriei coefficientium differentiae secundae fiant constantes, tum ipsius seriei propositae erit

$$= \frac{x}{1-x}a + \frac{x^2}{(1-x)^2}\Delta a + \frac{x^3}{(1-x)^3}\Delta\Delta a.$$

Unde summae huiusmodi serierum ex differentiis coefficientium facile invenientur.

I. *Quaeratur summa huius seriei:*

$$1x + 3x^2 + 5x^3 + 7x^4 + 9x^5 + \text{\&c.}$$

Diff. I. 2, 2, 2, 2, &c.

Cum ergo differentiae primae sint constantes, ob $a = 1$ & $\Delta a = 2$: erit seriei propositae summa

$$= \frac{x}{1-x} + \frac{2xx}{(1-x)^2} = \frac{x+xx}{(1-x)^2}.$$

II. *Quaeratur summa huius seriei*

$$1x + 4xx + 9x^3 + 16x^4 + 25x^5 + \text{\&c.}$$

Diff. I. 3, 5, 7, 9, &c.

Diff. II. 2, 2, 2, &c.

Quia itaque est

$a = 1$; $\Delta a = 3$, $\Delta^2 a = 2$;

erit seriei propositae summa

$$= \frac{x}{1-x} + \frac{3xx}{(1-x)^2} + \frac{2x^3}{(1-x)^3} = \frac{x+xx}{(1-x)^3}.$$

III. *Quaeratur summa huius seriei*

$$S = 4x + 15x^2 + 40x^3 + 85x^4 + 156x^5 + 259x^6 + \&c.$$

Diff. I. 11, 25, 45, 71, 103

Diff. II. 14, 20, 26, 32

Diff. III. 6, 6, 6,

Quia est $a = 4$; $\Delta a = 11$; $\Delta^2 a = 14$; $\Delta^3 a = 6$; erit summa

$$S = \frac{4x}{1-x} + \frac{11xx}{(1-x)^2} + \frac{14x^3}{(1-x)^3} + \frac{6x^4}{(1-x)^4},$$

sive $S = \frac{4x - xx + 4x^3 - x^4}{(1-x)^4} = \frac{x(1+xx)(4-x)}{(1-x)^4}$.

5. Quanquam hoc modo istarum serierum in infinitum progredientium summae inveniuntur; tamen ex iisdem principiis hae series quoque ad datum quemvis terminum summari possunt. Proposita enim sit haec series

$$S = ax + bx^2 + cx^3 + dx^4 + \ldots\ldots\ldots\ldots + ox^n$$

& quaeratur eius summa, si in infinitum progrediatur, quae erit $= \frac{x}{1-x} a + \frac{xx}{(1-x)^2} \Delta a + \frac{x^3}{(1-x)^3} \Delta^2 a + \&c.$

Nunc considerentur eiusdem seriei termini post ultimum ox^n sequentes, qui sint

$$px^{n+1} + qx^{n+2} + rx^{n+3} + sx^{n+4} + \&c.$$

cuius seriei, si per x^n dividatur, summa, ut ante inveniri poterit; quae rursus per x^n multiplicata erit

$$\frac{x^{n+1}}{1-x} p + \frac{x^{n+2}}{(1-x)^2} \Delta p + \frac{x^{n+3}}{(1-x)^3} \Delta^2 p + \&c.$$

quae summa si a totius seriei in infinitum continuatae summa subtrahatur, remanebit summa portionis propositae quaesita: $S =$

$$\frac{x}{1-x}(a - x^n p) + \frac{x^2}{(1-x)^2}(\Delta a - x^n \Delta p) + \frac{x^3}{(1-x)^3}(\Delta^2 a - x^n \Delta^2 p)$$

&c.

I. *Quaeratur summa huius seriei finitae.*

$$S = 1x + 2x^2 + 3x^3 + 4x^4 + \ldots\ldots\ldots + nx^n$$

Tam

Tam horum coefficientium, quam terminum ultimum sequentium quaerantur differentiae:

$$\begin{array}{ccccc|cccc} 1, & 2, & 3, & 4, & \&c. & n+1, & n+2, & n+3, & \&c. \\ & 1, & 1, & 1, & & & 1, & 1; & \end{array}$$

eritque $a = 1$, $\Delta a = 1$, $p = n+1$, $\Delta p = 1$,
unde summa quaesita est;

$$S = \frac{x}{1-x}(1-(n+1)x^n) + \frac{x^2}{(1-x)^2}(1-x^n), \quad \text{seu}$$

$$S = \frac{x-(n+1)x^{n+1}+nx^{n+2}}{(1-x)^2}$$

II. *Quaeratur summa huius seriei finitae*

$$S = 1x + 4x^2 + 9x^3 + 16x^4 + \ldots\ldots + n^2x^n.$$

Investigentur primum differentiae hoc modo:

$$\begin{array}{ccccc|cccc} 1, & 4, & 9, & 16, & \&c. & (n+1)^2, & (n+2)^2, & (n+3)^2, & \&c. \\ & 3, & 5, & 7, & & & 2n+3, & 2n+5 & \\ & & 2, & 2, & & & & 2 & \end{array}$$

quibus inventis erit summa quaesita $S =$

$$\frac{x}{1-x}(1-(n+1)^2x^n) + \frac{x^2}{(1-x)^2}(3-(2n+3)x^n) + \frac{x^3}{(1-x)^3}(2-2x^n),$$

seu $$S = \frac{x+xx-(n+1)^2x^{n+1}+(2nn+2n-1)x^{n+2}-nnx^{n+3}}{(1-x)^3},$$

6. Quodsi autem series proposita non eiusmodi habeat coefficientes, qui tandem ad differentias constantes deducantur, tum transmutatio hic exhibita nihil confert ad eius summam determinandam. Neque vero etiam eius ope summa proxime definiri poterit commodius, quam per ipsam seriei propositae additionem fieri licet. Si enim in serie

$$ax + bx^2 + cx^3 + dx^4 + \&c.$$

fuerit $x < 1$ quo solo casu summatio proprie sic dicta locum habere potest, erit $\frac{x}{1-x} > x$, ideoque nova series minus convergit quam proposita. Sin autem in serie proposita fuerit

$x = 1$

$x = 1$ tum novae feriei omnes plane termini fiunt infiniti, quo ergo cafu ifta transfmutatio nullius prorfus erit ufus.

7. Confideremus autem feriem, in qua figna + & — alternatim fe excipiant, quae ex praecedente deducetur ponendo x negativum. Si itaque fuerit

$$S = ax - bx^2 + cx^3 - dx^4 + ex^5 - \&c.$$

cuius feriei negativa oritur, fi in praecedente ftatuatur x negativum. Sumantur ergo ut ante differentiae Δa, $\Delta^2 a$, $\Delta^3 a$, &c. ex ferie coefficientium a, b, c, d, e, &c. fignis ad folas ipfius x poteftates relatis, atque feries propofita transformabitur in hanc: $S =$

$$\frac{x}{1+x}a - \frac{x^2}{(1+x)^2}\Delta a + \frac{x^3}{(1+x)^3}\Delta^2 a - \frac{x^4}{(1+x)^4}\Delta^3 a + \&c.$$

unde perfpicitur aequationem propofitam iifdem cafibus fummari poffe quibus praecedens. Scilicet fi feries a, b, c, d, &c. tandem ad differentias conftantes deducatur.

8. Hoc autem cafu ifta transformatio commodam praebet approximationem ad valorem feriei propofitae:

$$ax - bx^2 + cx^3 - dx^4 + ex^5 - fx^6 + \&c.$$

quantufcunque enim x fuerit numerus, fractio $\frac{x}{1+x}$ fecundum cuius poteftates altera feries progreditur, fit unitate minor: atque fi fit $x = 1$, erit $\frac{x}{1+x} = \frac{1}{2}$. Sin autem fit $x < 1$ puta $x = \frac{1}{n}$ fiet $\frac{x}{1+x} = \frac{1}{n+1}$, ideoque feries per transformationem inventa femper magis convergit quam propofita. Confideremus imprimis cafum, quo $x = 1$, qui ad feries fummandas ingens affert fubfidium, fitque

$$S = a - b + c - d + e - f + \&c.$$

ac denotentur differentiae primae, fecundae & fequentes ipfius a, quas progreffio a, b, c, d, e, &c. praebet per

Δ,

Δa, $\Delta^2 a$, $\Delta^3 a$, &c. quibus inventis erit

$$S = \tfrac{1}{2}a - \tfrac{1}{4}\Delta a + \tfrac{1}{8}\Delta^2 a - \tfrac{1}{16}\Delta^3 a + \&c.$$

quae nisi actu terminatur, summam vero proximam satis commode exhibet.

9. Usum igitur huius ultimae transmutationis, qua sumsimus $x = 1$, in aliquot exemplis ostendamus, ac primo quidem in eiusmodi, quibus vera summa finite exprimi potest. Tales sunt series divergentes, quibus numeri a, b, c, d, &c. tandem ad differentias constantes deducant, quarum summae, cum recepto huius vocis significatu, proprie non exhiberi queant, vocem summae hic eo sensu, quem supra tribuimus, accipimus, ita ut denotet valorem expressionis finitae, ex cuius evolutione proposita series nascatur.

I. *Sit igitur proposita haec series Leibnitzii:*

$$S = 1 - 1 + 1 - 1 + 1 - 1 + \&c.$$

in qua cum omnes termini sint aequales, fient omnes differentiae $= 0$, ideoque ob $a = 1$, erit $S = \tfrac{1}{2}$.

II. *Sit proposita ista series:*

$$S = 1 - 2 + 3 - 4 + 5 - 6 + \&c.$$

Diff. I. $= 1,\ 1,\ 1,\ 1,\ 1,$ &c.

Cum ergo sit $a = 1$, $\Delta a = 1$, erit $S = \tfrac{1}{2} - \tfrac{1}{4} = \tfrac{1}{4}$.

III. *Sit proposita haec series:*

$$S = 1 - 3 + 5 - 7 + 9 - \&c.$$

Diff. I. $= 2,\ 2,\ 2,\ 2,$ &c.

Ob $a = 1$ & $\Delta a = 2$ fit $S = \tfrac{1}{2} - \tfrac{2}{4} = 0$.

IV. *Sit proposita haec series trigonalium numerorum.*

$$S = 1 - 3 + 6 - 10 + 15 - 21 + \&c.$$

Diff. I. $= 2,\ 3,\ 4,\ 5,\ 6,$ &c.

Diff. II. $= 1,\ 1,\ 1,\ 1,$ &c.

Hic ergo ob $a = 1$, $\Delta a = 2$, & $\Delta\Delta a = 1$; erit

$$S = \tfrac{1}{2} - \tfrac{2}{4} + \tfrac{1}{8} = \tfrac{1}{8}.$$

V. *Sit proposita series quadratorum:*

$$S = 1 - 4 + 9 - 16 + 25 - 36 + \&c.$$

Diff. I. $= 3,\ 5,\ 7,\ 9,\ 11,$ &c.

Diff. II. $= 2,\ 2,\ 2,\ 2,$ &c.

Ob

Ob $a = 1$; $\Delta a = 3$; $\Delta\Delta a = 2$; erit $S = \frac{1}{2} - \frac{3}{4} + \frac{2}{8} = 0$.

VI. *Sit propofita feries biquadratorum:*

$$S = 1 - 16 + 81 - 256 + 625 - 1296 + \&c.$$

Diff. I. $= 15,\ 65,\ 175,\ 369,\ 671$

Diff. II. $= 50,\ 110,\ 194,\ 302$

Diff. III. $= 60,\ 84,\ 108$

Diff. IV. $= 24,\ 24$

Erit ergo $S = \frac{1}{2} - \frac{15}{4} + \frac{50}{8} - \frac{60}{16} + \frac{24}{32} = 0$.

10. Si feries magis divergant uti geometricae aliaeque fimiles, eae hoc modo ftatim in feriem magis convergentem tranfmutantur, quae nifi adhuc fatis convergat, eodem modo in aliam magis convergentem convertetur.

I. *Sit propofita haec feries geometrica:*

$$S = 1 - 2 + 4 - 8 + 16 - 32 + \&c.$$

Diff. I. $= 1,\ 2,\ 4,\ 8,\ 16,\ \&c.$

Diff. II. $= 1,\ 2,\ 4,\ 8,\ \&c.$

Diff. III. $= 1,\ 2,\ 4,\ \&c.$

Cum igitur in omnibus differentiis primus terminus fit $= 1$. Summa feriei exprimetur hoc modo

$$S = \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \frac{1}{32} - \frac{1}{64} + \&c.$$

cuius fumma eft $= \frac{1}{3}$, oritur enim ex evolutione fractionis $\frac{1}{2+1}$, dum propofita oritur ex $\frac{1}{1+2}$.

II. *Sit propofita haec feries recurrens:*

$$S = 1 - 2 + 5 - 12 + 29 - 70 + 169 - \&c.$$

Diff. I. $= 1,\ 3,\ 7,\ 17,\ 41,\ 99\ \&c.$

Diff. II. $= 2,\ 4,\ 10,\ 24,\ 58\ \&c.$

Diff. III. $= 2,\ 6,\ 14,\ 34\ \&c.$

Diff. IV. $= 4,\ 8,\ 20\ \&c.$

Diff. V. $= 4,\ 12\ \&c.$

Diff. VI. $= 8\ \&c.$

&c.

Continuarum ergo differentiarum termini primi conftituunt hanc progreffionem geometricam geminatam:

1, 1, 2, 2, 4, 4, 8, 8, 16, 16, &c. unde erit

$$S = \frac{1}{2} - \frac{1}{4} + \frac{2}{8} - \frac{2}{16} + \frac{4}{32} - \frac{4}{64} + \frac{8}{128} \text{ \&c.}$$

cum igitur praeter primum terminum reliqui bini ſe continuo deſtruant, erit $S = \frac{1}{2}$. Oritur autem ſeries propoſita ex evolutione fractionis $\frac{1}{1+2-1} = \frac{1}{2}$, uti in expreſſione naturae ſerierum recurrentium oſtendimus.

III. *Sit propoſita ſeries hypergeometrica:*

$$S = 1 - 2 + 6 - 24 + 120 - 720 + 5040 - \text{\&c.}$$

cuius differentias continuas hoc modo commodius inveſtigabimus:

	Diff. I.	Diff. II.	Diff. III.
1			
2	1		
6	4	3	
24	18	14	11
120	96	78	64
720	600	504	426
5040	4320	3720	3216 &c.
40320	35280	30960	27240
362880	322560	287280	256320
3628800	3265920	2943360	2656080

Quibus differentiis ulterius continuatis erit:

$$S = \frac{1}{2} - \frac{1}{4} + \frac{3}{8} - \frac{11}{16} + \frac{53}{32} - \frac{309}{64} + \frac{2119}{128} - \frac{16687}{256}$$

$$+ \frac{148329}{512} - \frac{1468457}{1024} + \frac{16019531}{2048} - \frac{190899411}{4096} + \text{\&c.}$$

Colligantur duo termini initiales, eritque $S = \frac{1}{4} + A$

exiſtente $A = \frac{3}{8} - \frac{11}{16} + \frac{53}{32} - \frac{309}{64} + \frac{2119}{128} - \text{\&c.}$

Si nunc eodem modo differentiae capiantur, erit

 A

$$A = \frac{3}{2^4} - \frac{5}{2^6} + \frac{21}{2^8} - \frac{99}{2^{10}} + \frac{615}{2^{12}} - \frac{4401}{2^{14}} + \frac{36585}{2^{16}}$$
$$- \frac{342207}{2^{18}} + \frac{3565323}{2^{20}} - \frac{40866525}{2^{22}} + \&c.$$

Colligantur duo termini initiales, quia convergunt, fietque

$$A = \frac{7}{2^6} + B \quad \text{existente} \quad B = \frac{21}{2^8} - \frac{99}{2^{10}} + \&c.$$

cuius seriei differentiis denuo sumendis fiet:

$$B = \frac{21}{2^9} - \frac{15}{2^{12}} + \frac{159}{2^{15}} - \frac{429}{2^{18}} + \frac{5241}{2^{21}} - \frac{26283}{2^{24}} + \frac{338835}{2^{27}}$$
$$- \frac{2771097}{2^{30}} + \&c.$$

Colligantur quatuor termini initiales in unum & statuatur

$$B = \frac{153}{2^{12}} + \frac{843}{2^{18}} + C \quad \text{existente} \quad C = \frac{5241}{2^{21}} - \frac{26283}{2^{24}} + \&c.$$

fietque aliquot terminis actu colligendis proxime:

$$C = \frac{15645}{2^{24}} - \frac{60417}{2^{30}}.$$ Ex his ergo tandem concludetur summa seriei propositae: $S = 0,40082038$, quae tamen vix ultra tres quatuorve figuras pro accurata haberi potest ob nimiam seriei divergentiam, est tamen certe iusto minor. Aliunde enim inveni hanc summam esse $=$ $0,4036524077$, ubi ne ultima quidem nota a vero aberrat.

11. Imprimis autem haec transmutatio ingentem affert utilitatem ad series iam quidem, sed lente convergentes in alias, quae multo promtius convergant, transmutandas. Quoniam vero termini sequentes minores sunt quam praecedentes, differentiae primae fiunt negativae; unde in sequentibus signorum ratio probe est habenda.

I. *Sit proposita haec series:*

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \&c.$$

Diff.

Diff. I. $= -\frac{1}{2}; -\frac{1}{2.3}; -\frac{1}{3.4}; -\frac{1}{4.5}; -\frac{1}{5.6}$

Diff. II. $= +\frac{1}{3}; \frac{2}{2.3.4}; \frac{2}{3.4.5}; \frac{2}{4.5.6}$

Diff. III. $= -\frac{1}{4}; -\frac{2.3}{2.3.4.5}; -\frac{2.3}{3.4.5.6}$

Diff. IV. $= +\frac{1}{5}$; &c. Hinc ergo erit

$$S = \frac{1}{2} + \frac{1}{2.4} + \frac{1}{3.8} + \frac{1}{4.16} + \frac{1}{5.32} + \&c,$$

utramque autem hanc feriem logarithmum hyperbolicum binarii exhibere, iam in Introductione oftendimus.

II. *Sit propofita ifta feries pro circulo:*

$$S = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} \; \&c.$$

Diff. I. $= -\frac{2}{1.3}; -\frac{2}{3.5}; -\frac{2}{5.7}; -\frac{2}{7.9}; -\frac{2}{9.11}$ &c.

Diff. II. $= +\frac{2.4}{1.3.5}; \frac{2.4}{3.5.7}; \frac{2.4}{5.7.9}; \frac{2.4}{7.9.11}$ &c.

Diff. III. $= -\frac{2.4.6}{1.3.5.7}; -\frac{2.4.6}{3.5.7.9}; -$ &c.

&c.

Hinc ergo concluditur fore fummam feriei:

$$S = \frac{1}{2} + \frac{1}{3.2} + \frac{1.2}{3.5.2} + \frac{1.2.3}{3.5.7.2} + \&c.$$

feu $2S = 1 + \frac{1}{3} + \frac{1.2}{3.5} + \frac{1.2.3}{3.5.7} + \frac{1.2.3.4}{3.5.7.9} + \&c.$

III. *Quaeratur valor huius feriei infinitae*

$$S = l2 - l3 + l4 - l5 + l6 - l7 + l8 - l9 \; \&c.$$

Quia differentiae ab initio nimis fiunt inaequales, colligantur actu termini usque ad $l10$ ex tabulis, quorum valor reperietur $= -0,3911005$; eritque

$$S = -0,3911005 + l10 - l11 + l12 - l13 + l14 - l15 + \&c.$$
in infinitum.

Desumantur hi logarithmi ex tabulis, eorumque differentiae quaerantur hoc modo

	Diff. 1.	Diff. 2.	Diff. 3.	Diff. 4.	Diff. 5.
	+	—	+	—	+
$l10 = 1,0000000$					
$l11 = 1,0413927$	413927	36042			
$l12 = 1,0791812$	377885	30263	5779		
$l13 = 1,1139434$	347622	25776	4487	1292	368
$l14 = 1,1461280$	321846	22213	3563	924	
$l15 = 1,1760913$	299633				

Ex quibus reperitur

$$l10 - l11 + l12 - l13 + \&c. =$$

$$\frac{1,0000000}{2} - \frac{413927}{4} - \frac{36042}{8} - \frac{5779}{16} - \frac{1292}{32} - \frac{368}{64}$$

$$= 0,4891606.$$

Hinc valor seriei propositae erit

$$S = l2 - l3 + l4 - l5 + \&c. = 0,0980601;$$

cui logarithmo respondet numerus $1,253315$.

12. Quemadmodum has transmutationes obtinuimus ponendo in serie loco x hanc fractionem $\frac{y}{1 \pm y}$, ita innumerabiles aliae transmutationes orientur, si loco x aliae functiones ipsius y substituantur. Sit iterum proposita ista series:

$$S = ax + bx^2 + cx^3 + dx^4 + ex^5 + fx^6 + \&c.$$

atque ponatur $x = y(1 - y)$, quo facto series orietur sequens

$$\begin{aligned} S = ay &- ayy \\ &+ byy - 2by^3 + by^4 \\ &\quad + cy^3 - 3cy^4 + 3cy^5 - cy^6 \\ &\quad\quad + dy^4 - 4dy^5 + 6dy^6 \\ &\quad\quad\quad + ey^5 - 5ey^6 \quad \&c. \\ &\quad\quad\quad\quad + fy^6 \end{aligned}$$

Quodsi ergo altera harum serierum fuerit summabilis, simul alterius summa erit cognita. Ita si statuatur

$S = x + x^2 + x^3 + x^4 + x^5 + \&c. = \frac{x}{1-x}$, erit

$S = y - y^3 - y^4 + y^6 + y^7 - y^9 - y^{10} + \&c.$

Cuius ergo seriei summa erit $= \frac{y - yy}{1 - y + yy}$.

13. Si altera series alicubi abrumpatur, tum summa prioris absolute exhiberi poterit. Ponamus esse $a = 1$, & in serie inventa omnes terminos post primum evanescere, ut sit $S = y$; ideoque ob $x = y - yy$, erit summa prioris $= \frac{1}{2} - \sqrt{(\frac{1}{4} - x)}$. Fiet autem ob $a = 1$; ut sequitur:

$$b = \quad 1 = \frac{1}{4} . 2^2$$

$$c = \quad 2 = \frac{1 . 3}{4 . 6} . 2^4$$

$$d = \quad 5 = \frac{1 . 3 . 5}{4 . 6 . 8} . 2^6$$

$$e = \quad 14 = \frac{1 . 3 . 5 . 7}{4 . 6 . 8 . 10} . 2^8$$

$$f = \quad 42 = \frac{1 . 3 . 5 . 7 . 9}{4 . 6 . 8 . 10 . 12} . 2^{10}$$

$$g = 132 = \frac{1 . 3 . 5 . 7 . 9 . 11}{4 . 6 . 8 . 10 . 12 . 14} . 2^{12}$$

&c.

unde prior series abibit in hanc: $S = \frac{1}{2} - \sqrt{(\frac{1}{4} - x)} = x + x^2 + 2x^3 + 5x^4 + 14x^5 + 42x^6 + 132x^7 + \&c.$ quae eadem invenitur, si quantitas surda $\sqrt{(\frac{1}{4} - x)}$ in seriem evolvatur, atque ab $\frac{1}{2}$ subtrahatur.

14. Statuamus, quo transmutatio latius pateat $x = y(1 + ny)^r$, atque series proposita:

$$S = ax + bx^2 + cx^3 + dx^4 + ex^5 + \&c.$$

transmutabitur in sequentem: S =

$$\begin{array}{rlllll}
S = & ay + \frac{\nu}{1} nay^2 & + \frac{\nu(\nu-1)}{1.2} n^2 ay^3 & + \frac{\nu(\nu-1)(\nu-2)}{1.2.3} n^3 ay^4 & + \frac{\nu(\nu-1)(\nu-2)(\nu-3)}{1.2.3.4} n^4 ay^5 \\
& + by^2 & + \frac{2\nu}{1} nby^3 & + \frac{2\nu(2\nu-1)}{1.2} n^2 by^4 & + \frac{2\nu(2\nu-1)(2\nu-2)}{1.2.3} n^3 by^5 \\
& & + cy^3 & + \frac{3\nu}{1} ncy^4 & + \frac{3\nu(3\nu-1)}{1.2} n^2 cy^5 \\
& & & + dy^4 & + \frac{4\nu}{1} ndy^5 \\
& & & & + ey^5
\end{array}$$

&c.

Si ergo illius feriei fumma fuerit cognita, & huius fimul fumma habebitur, ac viciffim. Quoniam vero n & ν pro lubitu accipi poffunt, hinc ex una ferie fummabili innumerae aliae fummabiles inveniri poffunt.

15. Poffunt etiam eiufmodi tranfmutationes fieri, ut feriei inventae fumma fiat irrationalis, hoc modo.

Sit propofita ifta feries:

$$S = ax + bx^3 + cx^5 + dx^7 + ex^9 + fx^{11} + \&c. \quad \text{erit}$$
$$Sx = ax^2 + bx^4 + cx^6 + dx^8 + ex^{10} + fx^{12} + \&c.$$

Iam ftatuatur $x = \frac{y}{\sqrt{(1-nyy)}}$; erit $xx = \frac{yy}{1-nyy}$,

atque feries propofita tranfmutabitur in hanc: $\frac{Sy}{\sqrt{(1-nyy)}} =$

$$\begin{array}{llllll}
ay^2 + nay^4 & + n^2ay^6 & + n^3ay^8 & + n^4ay^{10} & + \&c. \\
+ by^4 & + 2nby^6 & + 3n^2by^8 & + 4n^3by^{10} & + \&c. \\
& + cy^6 & + 3ncy^8 & + 6n^2cy^{10} & + \&c. \\
& & + dy^8 & + 4ndy^{10} & + \&c. \\
& & & + ey^{10} & + \&c.
\end{array}$$

Si igitur fumma S ex priori ferie fuerit cognita, habebitur fimul fumma fequentis feriei: $\frac{S}{\sqrt{(1-nyy)}} =$

$$ay + (na+b)y^3 + (n^2a + 2nb + c)y^5 + (n^3a + 3n^2b + 3nc + d)y^7 + \&c.$$

16.

16. Si ſumatur $n = -1$; erunt coefficientes huius ſeriei differentiae continuae ipſius a, ex ſerie a, b, c, d, &c. ſin autem ſigna in ſerie propoſita alternentur, tum poſito $n = 1$ coefficientes erunt iſtae differentiae. Denotent ergo Δa, $\Delta^2 a$, $\Delta^3 a$, $\Delta^4 a$, &c. differentias primas, ſecundas, tertias, &c. ipſius a ex ſerie numerorum a, b, c, d, e, &c. Ac ſi fuerit

$$S = ax + bx^3 + cx^5 + dx^7 + ex^9 + \&c.$$

poſito $x = \frac{y}{\sqrt{(1+yy)}}$; erit

$$\frac{S}{\sqrt{(1+yy)}} = ay + \Delta a . y^3 + \Delta^2 a . y^5 + \Delta^3 a . y^7 + \&c.$$

Sin autem fuerit:

$$S = ax - bx^3 + cx^5 - dx^7 + ex^9 - \&c.$$

ponaturque $x = \frac{y}{\sqrt{(1-yy)}}$; erit

$$\frac{S}{\sqrt{(1-yy)}} = ay - \Delta a . y^3 + \Delta^2 a . y^5 - \Delta^3 a . y^7 + \&c.$$

Quodſi ergo ſeries a, b, c, d, e, &c. tandem ad differentias conſtantes deducat, tum utraque ſeries abſolute ſummari poterit; quae ſummatio autem quoque ex ſuperioribus ſequitur.

17. Ponamus coefficientes a, b, c, d, &c. conſtituere hanc ſeriem $1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}$, &c. eritque, uti ſupra iam vidimus:

$$a = 1$$

$$\Delta a = -\frac{2}{3}$$

$$\Delta^2 a = \frac{2.4}{3.5}$$

$$\Delta^3 a = -\frac{2.4.6}{3.5.7} \quad \&c.$$

unde sequentes duas series summabimus:

I. Sit $S = x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \frac{1}{7}x^7 + \&c.$

Erit $S = \frac{1}{2} l \frac{1+x}{1-x}$. Posito iam $x = \frac{y}{\sqrt{(1+yy)}}$, fiet

$$S = \frac{1}{2} l \frac{\sqrt{(1+yy)}+y}{\sqrt{(1+yy)}-y} = l(\sqrt{(1+yy)}+y):$$

unde erit

$$\frac{l(\sqrt{(1+yy)}+y)}{\sqrt{(1+yy)}} = y - \frac{2}{3}y^3 + \frac{2.4}{3.5}y^5 - \frac{2.4.6}{3.5.7}y^7 + \&c.$$

II. Sit $S = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \&c.$

Erit $S = A \text{ tang } x$. Posito iam $x = \frac{y}{\sqrt{(1-yy)}}$, fiet

$$S = A \text{ tang } \frac{y}{\sqrt{(1-yy)}} = A \sin y = A \cos \sqrt{(1-yy)}.$$

Hancobrem obtinebitur ista summatio:

$$\frac{A \sin y}{\sqrt{(1-yy)}} = y + \frac{2}{3}y^3 + \frac{2.4}{3.5}y^5 + \frac{2.4.6}{3.5.7}y^7 + \&c.$$

18. Possunt quoque loco x functiones transcendentes ipsius y substitui, sicque summationes aliae inventu difficiliores erui; verumtamen ne series novae fiant nimis perplexae, eiusmodi functiones eligi debent, quarum potestates facile exhiberi queant; quales sunt quantitates exponentiales e^y. Proposita igitur hac serie:

$$S = ax + bx^2 + cx^3 + dx^4 + ex^5 + fx^6 + \&c.$$

ponatur $x = e^{ny}y$, denotante e numerum, cuius logarithmus hyperbolicus $= 1$,

erit $x^2 = e^{2ny}y^2$; $x^3 = e^{3ny}y^3$; &c.

Generaliter vero est, uti constat:

$$e^z = 1 + z + \frac{z^2}{1.2} + \frac{z^3}{1.2.3} + \frac{z^4}{1.2.3.4} + \&c.$$

Quare series proposita in hanc transmutabitur:

$$\begin{array}{rl} S = ay + 1nay^2 + \tfrac{1}{2}n^2ay^3 + \tfrac{1}{6}n^3ay^4 + \tfrac{1}{24}n^4ay^5 + \&c. \\ + by^2 + \tfrac{2}{1}nby^3 + \tfrac{4}{2}n^2by^4 + \tfrac{8}{6}n^3by^5 + \&c. \\ + cy^3 + \tfrac{3}{1}ncy^4 + \tfrac{9}{2}n^2cy^5 + \&c. \\ + dy^4 + \tfrac{4}{1}ndy^5 + \&c. \\ + ey^5 + \&c. \end{array}$$

I. Sit ſeries propoſita geometrica:

$S = x + x^2 + x^3 + x^4 + x^5 + \&c.$ erit $S = \frac{x}{1 - x}$.

Ponatur iam

$n = -1$; ut ſit $x = e^{-y}y$; & $S = \frac{e^{-y}y}{1-e^{-y}y} = \frac{y}{e^y - y}$, reperietur ſumma haec

$$\frac{y}{e^y - y} = y - \frac{1}{2}y^3 - \frac{1}{6}y^4 + \frac{5}{24}y^5 + \frac{19}{120}y^6 - \&c.$$

cuius autem ſeriei lex non perſpicitur.

II. Sint in altera ſerie omnes termini praeter primum $= 0$; erit $b = -na$; $c = \frac{3}{2}n^2a$; $d = -\frac{8}{3}n^3a$;

$$e = \frac{125}{24}n^4a; \quad f = -\frac{29}{30}n^5a; \quad \&c.$$

Cum ergo ſit ſumma $S = ay$; & $x = e^{ny}$; fiet:

$$y = x - nx^2 + \frac{3}{2}n^2x^3 - \frac{8}{3}n^3x^4 + \frac{125}{24}n^4x^5 - \frac{29}{30}n^5x^6 + \&c.$$

Quoniam vero in his ſeriebus lex progreſſionis non eſt manifeſta, ſummationes ex hac ſubſtitutione deductae parum habent utilitatis. Praecipue autem notari merentur transformationes ex ſubſtitutione $x = \frac{y}{1 \pm y}$ derivatae, quippe quae non ſolum eximias ſummationes, ſed etiam idoneos modos ad ſummas ſerierum appropinquandi ſuppeditant. His ergo, quae ſine calculi differentialis ope ſunt expedita, praemiſſis, ad ipſum huius calculi uſum in doctrina ſerierum oſtendendum progrediamur.