			Log coeff.
$\epsilon = \epsilon$	—	$2'' \cos r$	0, 30
	—	$48 \cos 2\eta$	1, 681
	+	$11 \cos(2\eta - r)$	1, 041
	+	$3 \cos(2\eta + r)$	0, 48
	+	$36 \cos(2\Phi - 2\pi)$	1, 556
	+	$9 \cos(2\Phi - 2\pi - r)$	0, 95
	+	$3 \cos(2\Phi - 2\pi + r)$	0, 48
	+	$23 \cos(2\Phi - 2\pi - 2r)$	1, 362
	+	$484 \cos(2\theta - 2\pi)$	2, 6848
	+	$9 \cos(4\theta - 4\pi)$	0, 95
	—	$5 \cos(2\theta - 2\pi - r)$	0, 70
	+	$5 \cos(2\theta - 2\pi - + r)$	0, 70
	—	$7 \cos(2\theta - 2\pi - s)$	0, 84
	—	$3 \cos(2\theta - 2\pi + s)$	0, 48

Tabula autem pro diſtantia lunae a terra, vnde eius parallaxis et diameter apparens definiatur, ex formulis ſupra exhibitis facile conſtruetur.

ADDI-

ıde eius formulis

ıDDI-

ADDITAMENTUM
CONTINENS ALIAS METHODOS
INUESTIGANDI MOTUS LUNAE INAEQUALITATES.

Qui methodum ante deſcriptam accuratius euoluerit, eam quidem in ſe ſpectatam ſatis bonam atque plerisque lunae inaequalitatibus definiendis aptam deprehendet; interim tamen fateri cogor, eam non ſolum maxime eſſe operoſam, ſed etiam ita comparatam, vt plures inaequalitates, quae tamen motum lunae imprimis afficere videntur, non ſatis exacte exhibeat, et quaſi in dubio relinquat. Cauſa huius incertitudinis manifeſto in hoc eſt ſita, quod omnes inaequalitates ita inter ſe ſunt connexae, vt nullius valor verus accurate definiri poſſit, quin ſimul reliquae inaequalitates omnes fuerint cognitae. Cum igitur eiusmodi methodo approximandi ſim vſus, vt primo quasdam inaequalitates tanquam cognitas aſſumſerim, ex quibus deinceps reliquas definiuerim, probe notandum eſt ab his inuentis iterum priores, quae erant aſſumtae, leuem quandam mutationem pati; quae ſi ſtatim ab initio nota fuiſſet, etiam reliquarum valores aliquantillum mutati prodiiſſent: at quaedam inaequalitates adeo ſunt lubricae, vt facta vel minima mutatione in iis, a quibus pendent, inde non exiguam alterationem trahant. Huc imprimis pertinet motus apogei, cuius inueſtigatio omnes omnino inaequa-

 litates

litates implicat, ita vt fine harum cognitione neutiquam accurate definiri queat.

Cum igitur haec methodus iftis tantis incommodis fit obnoxia, aliam maxime diuerfam tentaui viam, quae ab iis effet libera, etiamfi negare nequeam, etiam hanc fuis non carere incommodis, quae tamen prorfus alius funt generis. Ex quo confido his duabus diuerfis methodis combinandis haud exiguum fructum in veram motuum lunarium cognitionem effe redundaturum. Praecipuum autem difcrimen verfatur in electione anomaliae, quae in fuperiore methodo non ita eft affumta, vt diftantia lunae a terra fieret vel maxima vel minima, fi anomalia vel $=0$ vel $=180°$ ftatuatur: neque enim differentiale diftantiae dx euanefcit, quando finus anomaliae in nihilum abit, fed praeterea etiamnunc ab elongatione folis a luna feu angulo η pendet. Ita fecundum hanc methodum neque apogaeum lunae neque perigaeum ibi ftatuitur, ubi angulus, quem motus lunae directio cum radio vectore facit, eft rectus; fed plerumque in alia puncta incidunt, quae ab iis locis, vbi luna terrae vel eft proxima, vel ab ea maxime remota, notabiliter fint diuerfa. Etfi autem in hoc calculo non verae lineae abfidum pofitio confideratur, hinc tamen methodus minime vitiofa eft reputanda; propterea quod non eft quaeftio, quo nomine quaepiam orbitae lunaris puncta appellentur, dummodo cunctae inaequalitates recte exprimantur. Sed quoniam circa has ipfas inaequalitates nonnulla grauiora dubia funt orta, haud abs re fore arbitror, et alteram methodum hic proponere.

I. Sit

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Sit

I.

Sit igitur vt ante: Maffa folis $=\odot$; terrae $=\text{♁}$ et lunae $=☽$; atque vis attractiua terrae in diftantia d vt $\frac{1}{dd}-\frac{1}{bb}$; manente vi folis quadrato diftantiae exacte proportionali. Tum vero fit

Longitudo lunae $=\phi$; latitudo $=\psi$; et diftantia curtata $=x$

Longitudo folis $=\theta$; eiusque a terra diftantia $=y$

Longitudo nodi afcendentis lunae $=\pi$ et inclinatio ad eclipticam $=\varrho$

ac ponatur breuitatis ergo elongatio lunae a fole $\phi-\theta=\eta$ et diftantia $\sqrt{(xx\,\text{fec.}\,\psi^2-2xy\,\text{cof}\,\eta+yy)}=z$.

Quibus pofitis fupra §. 20. vidimus motum lunae his quatuor aequationibus contineri:

$$\text{I.}\quad 2dxd\phi+xdd\phi=-\tfrac{1}{2}dt^2.\odot\left(\frac{y}{z^3}-\frac{1}{yy}\right)\text{fin}\,\eta$$

$$\text{II.}\quad ddx-xd\phi^2=-\tfrac{1}{2}dt^2\,(\text{♁}+☽)\,\text{cof}\,\psi^3\left(\frac{1}{xx}-\frac{1}{bb}\right)$$
$$-\tfrac{1}{2}dt^2.\odot\left(\frac{x-y\,\text{cof}\,\eta}{z^3}+\frac{\text{cof}\,\eta}{yy}\right)$$

$$\text{III.}\quad d\pi=-\tfrac{1}{2}dt^2.\odot\left(\frac{y}{z^3}-\frac{1}{yy}\right)\frac{\text{fin}\,(\phi-\pi)\,\text{fin}\,(\theta-\pi)}{xd\phi}$$

$$\text{IV.}\quad dl\,\text{tang}\,\varrho=\frac{d\pi}{\text{tang}\,(\phi-\pi)},\ \text{et tang}\,\psi=\text{tg}\,\varrho\,\text{cof}(\phi-\pi)$$

vbi elementum temporis dt fumtum eft pro conftante.

 II. Qua-

II.

Quatenus hic motus solis ingreditur, is pro regulari atque regulis Kepleri conformi haberi poterit: habebimus ergo

$$2\,dy\,d\theta + y\,dd\theta = 0 \text{ et } ddy - y\,d\theta^2 = -\tfrac{1}{2}\,dt^2 . \frac{\odot + \text{♁}}{yy}$$

vnde si ponamus orbitae solaris: semiparametrum $= c$; excentricitatem $= e$ et anomaliam veram $= u$

erit $y = \frac{c}{1-e\cos u}$; $du = d\theta = \frac{dt}{yy}\sqrt{\tfrac{1}{2}c(\odot+\text{♁})}$

Sit a semiaxis transuersus orbitae solis, ac tempore $= t$ sol motu medio absoluat angulum $= \omega$, quo pro mensura temporis t vtamur; erit ergo $d\omega = \frac{dt}{aa}\sqrt{\tfrac{1}{2}a(\odot+\text{♁})}$

ideoque $\tfrac{1}{2}\,dt^2 = \frac{a^3\,d\omega^2}{\odot+\text{♁}}$. At est $a = \frac{c}{1-ee}$. Hinc ergo fit

$$du = d\theta = \frac{a\,d\omega}{yy}\sqrt{ac} = \frac{aa\,d\omega}{yy}\sqrt{(1-ee)} = \frac{d\omega(1-e\cos u)^2}{(1-ee)\sqrt{(1-ee)}}$$

sicque tam du quam $d\theta$ per elementum $d\omega$ loco temporis introductum expressimus. Quia autem massa solis $\odot$ massam terrae ♁ tam enormiter excedit, sine errore pro $\tfrac{1}{2}\,dt^2$ scribi poterit $\frac{a^3\,d\omega^2}{\odot}$, eruntque nostrae aequationes pro luna:

I. $2\,dx\,d\Phi + x\,dd\Phi = -a^3\,d\omega^2\left(\frac{y}{z^3} - \frac{1}{yy}\right)\sin\eta$

II. $ddx - x\,d\Phi^2 = -\frac{a^3(\text{♁}+☽)\,d\omega}{\odot}\cos\psi^3\left(\frac{1}{xx} - \frac{1}{bb}\right) - a^3\,d\omega^2\left(\frac{x - y\cos\eta}{z^3} + \frac{\cos\eta}{yy}\right)$

III. $d\pi$

ro regu- rit: ha- $\frac{\odot+\text{♁}}{yy}$ ano- $= u$ ore $= t$ o men- $(\odot+\text{♁})$ ergo fit $\cos u)^2$ $1-ee)$ tempo. solis $\odot$ ore pro uationes $x - \frac{1}{bb}$) III. $d\pi$

III. $d\pi = -\frac{a^3\,d\omega^2}{x\,d\Phi}\left(\frac{y}{z^3} - \frac{1}{yy}\right)\sin(\Phi-\pi)\sin(\theta-\pi)$

IV. $d.l\tan\rho = \frac{d\pi}{\text{tg}(\Phi-\pi)}$; atque ob $\text{tg}\,\psi = \text{tg}\,\rho\,\text{cs}(\Phi-\pi)$, habebitur proxime $\cos\psi^3 = 1 - \tfrac{3}{4}\,\text{tg}\,\rho^2 - \tfrac{3}{4}\,\text{tg}\,\rho^2\,\text{cs}\,2(\Phi-\pi)$.

III.

Incipiamus a duabus aequationibus prioribus, ac ponamus breuitatis gratia

$$a^3\left(\frac{y}{z^3} - \frac{1}{yy}\right)\sin\eta = M \text{ et}$$

$$\frac{a^3(\text{♁}+☽)}{\odot}\cos\psi^3\left(\frac{1}{xx} - \frac{1}{bb}\right) + a^3\left(\frac{x - y\cos\eta}{z^3} + \frac{\cos\eta}{yy}\right) = \frac{A}{xx} + N$$

quandoquidem haec posterior expressio terminum inuoluit formae $\frac{A}{xx}$ prae ceteris incomparabiliter maiorem; atque habebimus has duas aequationes:

$$2\,dx\,d\Phi + x\,dd\Phi = -M\,d\omega^2 \text{ et } ddx - x\,d\Phi^2 = -\frac{A\,d\omega^2}{xx} - N\,d\omega^2$$

quarum prior per $2x^3\,d\Phi$ multiplicata ob $d\omega$ constans habebit integrale:

$$x^4\,d\Phi^2 = -2\,d\omega^2 \int M x^3\,d\Phi$$

Tum prior multiplicata per $2x\,d\Phi$ addatur ad posteriorem per $2\,dx$ multiplicatam, eritque, aggregatum:

$$2x\,dx\,d\Phi^2 + 2xx\,d\Phi\,dd\Phi + 2\,dx\,ddx = -2Mx\,d\omega^2\,d\Phi - \frac{2A\,d\omega^2\,dx}{xx} - 2N\,d\omega^2\,dx$$

Cuius integrale erit:

$$dx^2 + xx\,d\Phi^2 = +\frac{2A\,d\omega^2}{x} - 2\,d\omega^2 \int(Mx\,d\Phi + N\,dx)$$

IV.

Ponantur formulae integrales, quae in his expressionibus insunt:

$$-\int Mx^3\, d\Phi = P \quad \text{et} \quad -\int (Mx\,d\Phi + N\,dx) = Q$$

vt habeamus has duas aequationes;

$$x^4\, d\Phi^2 = 2P\,d\omega^2 \quad \text{et} \quad dx^2 + xx\,d\Phi^2 = \frac{2A\,d\omega^2}{x} + 2Q\,d\omega^2$$

vnde cum sit $xx\,d\Phi^2 = \frac{2P\,d\omega^2}{xx}$ erit

$$dx^2 = 2d\omega^2\left(Q + \frac{A}{x} - \frac{P}{xx}\right) \quad \text{et} \quad dx = \pm d\omega \sqrt{2}\left(Q + \frac{A}{x} - \frac{P}{xx}\right)$$

sicque differentiale dx per $d\omega$ exprimitur. Deinde vero habetur

$$d\Phi = \frac{d\omega}{xx}\sqrt{2P}$$

estque per hypothesin:

$$dP = -Mx\,d\omega\sqrt{2P} \quad \text{et} \quad dQ = -\frac{M\,d\omega}{x}\sqrt{2P} \mp N\,d\omega\sqrt{2}\left(Q + \frac{A}{x} - \frac{P}{xx}\right)$$

vbi quidem signorum ambiguorum inferius locum habere statuamus, quia motum ab apogeo numerare in animo est, ita vt hinc exeundo distantia x minuatur.

V.

Cum igitur differentiale dx in apogeo et perigeo euanescat, necesse est vt his locis formula irrationalis $\sqrt{\left(Q + \frac{A}{x} - \frac{P}{xx}\right)}$ in nihilum abeat, in reliquis autem locis valorem sortiatur realem. Commodissime ergo haec formula per sinum cuiuspiam anguli v exhibebitur, qui cum in apogeo euanescat, in perigeo autem duobus rectis

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ctis aequalis fiat, anomaliam lunae referet: idque sensu vero, ita vt distantia x in apogeo prodeat maxima, in perigeo vero minima. Sit igitur vt formam motus regularis sequamur:

semilatus rectum orbitae lunaris $= p$
excentricitas orbitae $= q$
et anomalia vera lunae $= v$

eritque hinc per eandem legem distantia $x = \frac{p}{1 - q\cos v}$.

Verum hic quantitates p et q, quae in motu regulari essent constantes, nunc pro variabilibus sunt habendae, earumque variabilitas per variabilitatem quantitatum P et Q, quae in motu regulari itidem sunt constantes, determinari debebit.

VI.

Substituamus ergo valorem assumtum $x = \frac{p}{1 - q\cos v}$ in formula irrationali $\sqrt{\left(Q + \frac{A}{x} - \frac{P}{xx}\right)}$, quae abibit in

$$\frac{1}{p}\sqrt{(Qpp + Ap(1 - q\cos v) - P(1 - q\cos v)^2)}$$

et euoluta dabit

$$\frac{1}{p}\sqrt{(Qpp + Ap - P - Apq\cos v + 2Pq\cos v - Pqq\cos v^2)}$$

quae vt reducatur ad formam V sin v, statuatur

primo $2P - Ap = 0$
tum vero $Qpp + Ap - P = Pqq$

ac nostra formula fiet $= \frac{1}{p}\sqrt{Pqq \sin v^2} = \frac{q \sin v}{p}\sqrt{P}$, habebimusque

$$dx = -\frac{q\,d\omega\,\text{fin}\,v}{p}\sqrt{2P} \quad \text{et}$$

$$dQ = -\frac{M\,d\omega}{x}\sqrt{2P} + \frac{N\,q\,d\omega\,\text{fin}\,v}{p}\sqrt{2P}$$

VII.

Cum iam fit $2P - Ap = o$; erit $P = \frac{1}{2}Ap$: quo valore in altera formula fubftituto orietur:

$$Qpp + \tfrac{1}{2}Ap = \tfrac{1}{2}Apqq \quad \text{feu} \quad Q = -\frac{A}{2p}(1-qq)$$

Sumantur nunc differentialia; eritque $dP = -Mx\,d\omega\sqrt{2P} = \frac{1}{2}A\,dp$, quae ob $2P = Ap$ abit in hanc

$$-Mx\,d\omega\sqrt{Ap} = \tfrac{1}{2}A\,dp, \text{ fiue } dp = -\frac{2Mx\,d\omega}{A}\sqrt{Ap}$$

vel etiam $\sqrt{Ap} = -\int Mx\,d\omega$

Simili modo erit

$$dQ = +\frac{A\,dp(1-qq)}{2pp} + \frac{Aq\,dq}{p} = -\frac{Mx\,d\omega(1-qq)}{pp}\sqrt{Ap} + \frac{Aq\,dq}{p}$$

ideoque

$$\frac{Aq\,dq}{p} = M\,d\omega\left(\frac{x(1-qq)}{pp} - \frac{1}{x}\right)\sqrt{Ap} + \frac{Nq\,d\omega\,\text{fin}\,v}{p}\sqrt{Ap}$$

$$\text{At eft } \frac{x(1-qq)}{pp} - \frac{1}{x} = \frac{x}{pp}\left(1-qq-\frac{pp}{xx}\right) = \frac{x}{pp}(1-qq-1+2q\,\text{cof}\,v-qq\,\text{cof}\,v^2)$$

$$\text{fiue} \quad \frac{x(1-qq)}{pp} - \frac{1}{x} = \frac{qx}{pp}(2\,\text{cof}\,v - q - q\,\text{cof}\,v^2)$$

Hinc ergo colligitur:

$$dq = \frac{Mx\,d\omega}{Ap}(2\,\text{cof}\,v - q - q\,\text{cof}\,v^2)\sqrt{Ap} + \frac{N\,d\omega\,\text{fin}\,v}{A}\sqrt{Ap} \text{ fiue}$$

$$dq = d\omega\left(\frac{M}{A}\left(2\,\text{cof}\,v - \frac{q\,\text{fin}\,v^2}{1-q\,\text{cof}\,v}\right) + \frac{N}{A}\,\text{fin}\,v\right)\sqrt{Ap}$$

VIII.

VIII.

Inuenta iam relatione differentialium dx, dp et dq ad differentiale temporis $d\omega$ fcilicet:

$$dx = -\frac{q\,d\omega\,\text{fin}\,v}{q}\sqrt{Ap}; \quad dp = -\frac{2Mx\,d\omega}{A}\sqrt{Ap}$$

$$\text{et } dq = d\omega\left(\frac{M}{A}\left(2\,\text{cof}\,v - \frac{q\,\text{fin}\,v^2}{1-q\,\text{cof}\,v}\right) + \frac{N}{A}\,\text{fin}\,v\right)\sqrt{Ap}$$

fupereft, vt quoque relationem elementi anomaliae dv definiamus. Cum igitur fit

$$x = \frac{p}{1-q\,\text{cof}\,v}, \text{ erit } 1 - q\,\text{cof}\,v = \frac{p}{x}; \text{ hincque differentiando}$$

$$q\,dv\,\text{fin}\,v = dq\,\text{cof}\,v + \frac{dp}{x} - \frac{p\,dx}{xx};$$

fubftituantur valores pro dq, dp et dx inuenti; ac diuifione facta per $q\,\text{fin}\,v$ prodibit

$$dv = \frac{d\omega}{xx}\sqrt{Ap} - \frac{d\omega}{q}\left(\frac{M}{A}\left(2\,\text{fin}\,v + \frac{q\,\text{fin}\,v\,\text{cof}\,v}{1-q\,\text{cof}\,v}\right) - \frac{N}{A}\,\text{cof}\,v\right)\sqrt{Ap}$$

Pro elemento autem longitudinis $d\Phi$ ob $2P = Ap$, ex antecedentibus habemus:

$$d\Phi = \frac{d\omega}{xx}\sqrt{Ap} = \frac{d\omega(1-q\,\text{cof}\,v)^2}{pp}\sqrt{Ap}$$

IX.

Ex his formulis ftatim fe offert motus apogei; cum enim longitudo apogei fit $= \Phi - v$, erit eius differentiale pro tempufculo $d\omega$:

$$d\Phi - dv = \frac{d\omega}{q}\left(\frac{M}{A}\left(2\,\text{fin}\,v + \frac{q\,\text{fin}\,v\,\text{cof}\,v}{1-q\,\text{cof}\,v}\right) - \frac{N}{A}\,\text{cof}\,v\right)\sqrt{Ap}$$

cuius ergo integrale praebebit verum motum apogei cum omnibus inaequalitatibus, quibus perturbatur. Vnde qui-

quidem perſpicitur, quod per ſe eſt manifeſtum, ſi quantitates M et N euaneſcerent, motum apogei fore nullum, ſeu apogeum perpetuo in loco fixo eſſe permanſurum. Deinde etiam iuuabit notaſſe has formulas:

$$d.\, q \cos v = -q\, d\Phi \sin v + \frac{2M}{A}\, d\omega \sqrt{Ap}$$

$$d.\, q \sin v = +q\, d\Phi \cos v + d\omega\left(\frac{N}{A} - \frac{M}{A} \cdot \frac{q \sin v}{1 - q\cos v}\right)\sqrt{Ap}$$

Tandem quoque habemus ex motu ſolis $du = d\theta = \frac{d\omega\,(1 - e\cos u)^2}{(1-ee)\sqrt{(1-ee)}}$ ideoque

$$d\eta = d\Phi - d\theta = d\omega\left(\frac{(1-q\cos v}{pp}\sqrt{Ap} - \frac{(1 - e\cos u)^2}{(1-ee)\sqrt{(1-ee)}}\right)$$

X.

Inuentis nunc omnium differentialium relationibus ad elementum temporis $d\omega$, euoluamus valores litterarum M et N, ac primo quidem cum ſit $z = \sqrt{(yy - 2xy\cos\eta + xx \sec.\psi^2)}$; quoniam quantitas z nonniſi in terminis minimis occurrit, pro ſec. ψ tuto unitas ſcribi poterit, et quia y tantopere excedit x, erit proxime

$$\frac{1}{z^3} = \frac{1}{y^3} + \frac{3x}{y^4}\cos\eta + \frac{3xx}{2y^5}(5\cos\eta^2 - 1) \quad \text{ſiue}$$

$$\frac{1}{z^3} = \frac{1}{y^3} + \frac{3x}{y^4}\cos\eta + \frac{3xx}{4y^5}(3 + 5\cos 2\eta)$$

Ideoque hinc habebitur:

$$\frac{y}{z^3} - \frac{1}{yy} = \frac{3x}{y^3}\cos\eta + \frac{3xx}{4y^4}(3 + 5\cos 2\eta)$$

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Vnde obtinemus:

$$M = a^3\left(\frac{3x}{2y^3}\sin 2\eta + \frac{3xx}{8y^4}(\sin\eta + 5\sin 3\eta)\right)$$

$$N = \frac{a^3(\text{♁} + \text{☽})}{\odot}\cos\psi^3\left(\frac{1}{xx} - \frac{1}{bb}\right) - \frac{A}{xx}$$

$$- a^3\left(\frac{x}{2y^3}(1 + 3\cos 2\eta) + \frac{3xx}{8y^4}(3\cos\eta + 5\cos 3\eta)\right)$$

XI.

Cum ſit proxime $\cos\psi^3 = 1 - \frac{1}{2}\tan\rho^2 - \frac{1}{2}\tan\rho^2\cos 2(\Phi - \pi)$, eius valor vnitate erit minor, atque ex parte conſtante, et parte variabili conſtabit, quae illa multo erit minor. Ponatur ergo

$\cos\psi^3 = \lambda + \Pi$; vt ſit $\Pi = 1 - \lambda - \frac{1}{2}\tan\rho^2 - \frac{1}{2}\tan\rho^2\cos 2(\Phi - \pi)$

vbi λ denotat partem conſtantem vnitate proxime aequalem, Π vero partem variabilem.

Erit ergo:

$$N = \frac{\lambda a^3(\text{♁} + \text{☽})}{\odot}\left(\frac{1}{xx} - \frac{1}{bb}\right) - \frac{A}{xx} + \frac{a^3(\text{♁} + \text{☽})}{\odot}\Pi\left(\frac{1}{xx} - \frac{1}{bb}\right)$$

$$- a^3\left(\frac{x}{2y^3}(1 + 3\cos 2\eta) + \frac{3xx}{8y^4}(3\cos\eta + 5\cos 3\eta)\right)$$

Statuatur nunc $A = \frac{\lambda a^3(\text{♁} + \text{☽})}{\odot}$; vt fiat

$$N = -\frac{A}{bb} + A\Pi\left(\frac{1}{xx} - \frac{1}{bb}\right)$$

$$- a^3\left(\frac{x}{2y^3}(1 + 3\cos 2\eta) + \frac{3xx}{8y^4}(3\cos\eta + 5\cos 3\eta)\right)$$

ac

ac ponatur breuitatis gratia: $p = b(1+\xi)$

erit $\dfrac{\sqrt{Ap}}{pp} = \sqrt{\dfrac{A}{b^3(1+\xi)^3}} = (1 - \tfrac{3}{2}\xi + \tfrac{15}{8}\xi^2)\sqrt{\dfrac{A}{b^3}}$

ob ξ prae 1 vehementer paruum, ſitque porro:

$$\sqrt{\frac{A}{b^3}} = \sqrt{\frac{\lambda a^3(\text{♁}+\text{☽})}{\odot}} = m,$$

atque habebitur $d\Phi = m\,d\omega\,(1 - \tfrac{3}{2}\xi + \tfrac{15}{8}\xi^2)(1 - q\cos v)^2$

XII.

Subſtituantur nunc pro x et y valores $\dfrac{p}{1-q\cos v}$ et $\dfrac{c}{1-e\cos u}$, eritque

$$M = a^3\left(\frac{3p(1-e\cos u)^3}{2c^3(1-q\cos v)}\sin 2\eta + \frac{3pp(1-e\cos u)^4}{8c^4(1-q\cos v)^2}(\sin\eta + 5\sin 3\eta)\right)$$

$$N = -\frac{A}{bb} + A\Pi\left(\frac{(1-q\cos v)^2}{pp} - \frac{1}{bb}\right)$$

$$-a^3\left(\frac{p(1-e\cos u)^3}{2c^3(1-q\cos v)}(1+3\cos 2\eta) + \frac{3pp(1-e\cos u)^4}{8c^4(1-q\cos v)^2}(3\cos\eta + 5\cos 3\eta)\right)$$

vbi quidem quoque terminus $\dfrac{A\Pi}{bb}$ prae termino $\dfrac{A}{bb}$ omitti poteſt. Nunc vt hinc valores $\dfrac{M}{A}\sqrt{Ap}$ et $\dfrac{N}{A}\sqrt{Ap}$ commode exprimantur, erit

$$\frac{a^3 b}{Ac^3}\sqrt{Ab} = \frac{a^3}{mc^3} = \frac{1}{m(1-ee)^3} = \frac{1+3ee}{m}$$

quoniam in his terminis minimis pro $1-ee$ ſcribere licet 1. Tum vero ſit $\dfrac{b}{c} = n$, eritque n fractio valde parua.

XIII.

XIII.

Factis ergo his ſubſtitutionibus, ob $p = b(1+\xi)$ habebimus:

$$\frac{M}{A}\sqrt{Ap} = \frac{3(1+3ee)}{2m}\frac{(1-e\cos u)^3}{1-q\cos v}(1+\tfrac{1}{2}\xi)\sin 2\eta$$

$$+\frac{3n}{8m}\frac{(1-e\cos u)^4}{(1-q\cos v)^2}(1+\tfrac{5}{2}\xi)(\sin\eta + 5\sin 3\eta)$$

Pro altera valore $\dfrac{N}{A}\sqrt{Ap}$ ſtatuatur terminus minimus:

$$\frac{\sqrt{Ab}}{bb} = \sqrt{\frac{\lambda a^3 b(\text{♁}+\text{☽})}{\odot b^4}} = i\ ; \text{ eritque}$$

$$\frac{N}{A}\sqrt{Ap} = -\frac{(1+3ee)}{2m}\frac{(1-e\cos u)^3}{1-q\cos v}(1+\tfrac{3}{2}\xi)(1+3\cos 2\eta)$$

$$-\frac{3n}{8m}\frac{(1-e\cos u)^4}{(1-q\cos v)^2}(1+\tfrac{5}{2}\xi)(3\cos\eta + 5\cos 3\eta)$$

$$+m(1-q\cos v)^2(1-\tfrac{3}{2}\xi)\Pi - i$$

vbi notari oportet, terminos per n multiplicatos ratione praecedentium eſſe minimos; tum vero quantitates ξ et Π atque multo magis i eſſe fractiones prae vnitate fere euaneſcentes.

XIV.

Quoniam hi ipſi termini quantitates M et N inuoluentes ſunt valde parui, in iis ſine errore altiores poteſtates vtriusque excentricitatis q et e negligi posſunt. In terminis ergo primis ſimpliciter per m diuiſis excentricitates tantum ad duas dimenſiones introducanducantur, in terminis autem per $\frac{n}{m}$ multiplicatis penitus omittantur, quia fractio n iam fere quadrato excentricitatis q aequiualet. In termino autem littera minima Π affecto, quia is per numerum m fatis magnum, vtpote 13 fere, est multiplicatus, excentricitas q vnius dimensionis retineatur.

His obferuatis habebimus:

$$\frac{M}{A}\sqrt{}Ap = \begin{cases} +\frac{3}{2m}(1+\frac{3}{2}ee+\frac{1}{2}qq)\sin 2\eta + \frac{3q}{4m}\sin(2\eta-v) \\ +\frac{3q}{4m}\sin(2\eta+v) - \frac{9e}{4m}\sin(2\eta-u) - \frac{9e}{4m}\sin(2\eta+u) \\ +\frac{3qq}{8m}\sin(2\eta-2v) + \frac{3qq}{8m}\sin(2\eta+2v) \\ +\frac{9ee}{8m}\sin(2\eta-2u) + \frac{9ee}{8m}\sin(2\eta+2u) \\ -\frac{9eq}{8m}\sin(2\eta-v+u) - \frac{9eq}{8m}\sin(2\eta+v-u) \\ -\frac{9eq}{8m}\sin(2\eta-v-u) - \frac{9eq}{8m}\sin(2\eta+v+u) \\ +\frac{9}{4m}\xi\sin 2\eta + \frac{9}{8m}q\xi\sin(2\eta-v) + \frac{9}{8m}q\xi\sin(2\eta+v) \\ -\frac{27}{8m}e\xi\sin(2\eta-u) - \frac{27}{8m}e\xi\sin(2\eta+u) \\ +\frac{3n}{8m}\sin\eta + \frac{15n}{8m}\sin 3\eta \end{cases}$$

$\frac{N}{A}\sqrt{}Ap=$

$$\frac{N}{A}\sqrt{}Ap = \begin{cases} -\frac{1}{2m}(1+\frac{3}{2}ee+\frac{1}{2}qq) - \frac{3}{2m}(1+\frac{3}{2}ee+\frac{1}{2}qq)\cos 2\eta \\ -\frac{q}{2m}\cos v + \frac{3e}{2m}\cos u \\ -\frac{3q}{4m}\cos(2\eta-v) - \frac{3q}{4m}\cos(2\eta+v) \\ +\frac{9e}{4m}\cos(2\eta-u) + \frac{9e}{4m}\cos(2\eta+u) \\ -\frac{qq}{4m}\cos 2v + \frac{3eq}{4m}\cos(v-u) + \frac{3eq}{4m}\cos(v+u) \\ -\frac{3ee}{4m}\cos 2u \\ -\frac{3qq}{8m}\cos(2\eta-2v) - \frac{3qq}{8m}\cos(2\eta+2v) \\ -\frac{9ee}{8m}\cos(2\eta-2u) - \frac{9ee}{8m}\cos(2\eta+2u) \\ +\frac{9eq}{8m}\cos(2\eta-v+u) + \frac{9eq}{8m}\cos(2\eta+v-u) \\ +\frac{9eq}{8m}\cos(2\eta-v-u) + \frac{9eq}{8m}\cos(2\eta+v+u) \\ -\frac{3}{4m}\xi - \frac{9}{4m}\xi\cos 2\eta - \frac{3q}{8m}\xi\cos v \\ -\frac{9q}{8m}\xi\cos(2\eta-v) - \frac{9q}{8m}\xi\cos(2\eta+v) \\ +\frac{9e}{4m}\xi\cos u + \frac{27e}{8m}\xi\cos(2\eta-u) \\ +\frac{27e}{8m}\xi\cos(2\eta+u) - \frac{9n}{8m}\cos\eta - \frac{15n}{8m}\cos 3\eta \\ +m\Pi - 2mq\Pi\cos v - \frac{3}{2}m\xi\Pi - i \end{cases}$$

XV.

XV.

Quaeramus igitur valores euolutos noſtrorum differentialium ad elementum temporis applicatorum: ac primo quidem habebimus:

$$\frac{d\Phi}{d\omega} = m\left(1+\tfrac{3}{2}qq\right) - 2mq\cos v + \tfrac{1}{2}mqq\cos 2v - \tfrac{3}{2}m\left(1+\tfrac{3}{2}qq\right)\xi + 3mq\xi\cos v - \tfrac{3}{4}mqq\xi\cos 2v + \tfrac{15}{8}m\xi\xi$$

$$\frac{du}{d\omega} = \frac{d\theta}{d\omega} = 1 + 2ee - 2e\cos u + \tfrac{1}{2}ee\cos 2u;$$ vnde concludimus

$$\frac{d\eta}{d\omega} = m\left(1+\tfrac{3}{2}qq\right) - 1 - 2ee - 2mq\cos v + 2e\cos u + \tfrac{1}{2}mqq\cos 2v - \tfrac{1}{2}ee\cos 2u - \tfrac{3}{2}m\left(1+\tfrac{3}{2}qq\right)\xi + 3mq\xi\cos v - \tfrac{3}{4}mqq\xi\cos 2v + \tfrac{15}{8}m\xi\xi$$

Deinde cum ſit $\frac{dp}{d\omega} = -2x.\frac{M}{A}\sqrt{Ap} = -\frac{2b(1+\xi)}{1-q\cos v}.\frac{M}{A}\sqrt{Ap}$, ob $p = b(1+\xi)$ erit $\frac{d\xi}{d\omega} =$

$$\left(-2\left(1+\tfrac{1}{2}qq\right) - 2q\cos v - qq\cos 2v - 2\xi - 2q\xi\cos v\right)\frac{M}{A}\sqrt{Ap}$$

ac valorem pro $\frac{M}{A}\sqrt{Ap}$ inuentum ſubſtituendo obtinebimus ſequentes formulas:

$\frac{d\xi}{d\omega} =$

$$\frac{d\xi}{d\omega} = \begin{cases} -\frac{3}{m}\left(1+\tfrac{5}{2}ee+\tfrac{3}{2}qq\right)\sin 2\eta - \frac{3q}{m}\sin(2\eta-v) - \frac{3q}{m}\sin(2\eta+v) \\ +\frac{9e}{2m}\sin(2\eta-u) + \frac{9e}{2m}\sin(2\eta+u) \\ -\frac{9qq}{4m}\sin(2\eta-2v) - \frac{9qq}{4m}\sin(2\eta+2v) \\ -\frac{9ee}{4m}\sin(2\eta-2u) - \frac{9ee}{4m}\sin(2\eta+2u) \\ +\frac{9eq}{2m}\sin(2\eta-v+u) + \frac{9eq}{2m}\sin(2\eta+v-u) \\ +\frac{9eq}{2m}\sin(2\eta-v-u) + \frac{9eq}{2m}\sin(2\eta+v+u) \\ -\frac{15}{2m}\xi\sin 2\eta - \frac{15q}{2m}\xi\sin(2\eta-v) - \frac{15q}{2m}\xi\sin(2\eta+v) \\ +\frac{45e}{4m}\xi\sin(2\eta-u) + \frac{45e}{4m}\xi\sin(2\eta+u) \\ -\frac{3n}{4m}\sin\eta - \frac{15n}{4m}\sin 3\eta \end{cases}$$

XVI.

Porro cum ſit $\frac{q\sin v^2}{1-q\cos v} = \frac{q-q\cos 2v}{2(1-q\cos v)} = \tfrac{1}{2}q - \tfrac{1}{2}q\cos 2v + \tfrac{1}{4}qq\cos v - \tfrac{1}{4}qq\cos 3v$; erit

$$\frac{dq}{d\omega} = \left(2\cos v - \tfrac{1}{2}q + \tfrac{1}{2}q\cos 2v - \tfrac{1}{4}qq\cos v + \tfrac{1}{4}qq\cos 3v\right)\frac{M}{A}\sqrt{Ap} + \sin v.\frac{N}{A}\sqrt{Ap}$$

Facta ergo ſubſtitutione valorum pro $\frac{M}{A}\sqrt{Ap}$ et $\frac{N}{A}\sqrt{Ap}$ inuentorum, habebitur:

$\frac{dq}{d\omega} =$

$$\frac{d\eta}{d\omega} = \begin{cases}
+\frac{9}{4m}(1+\frac{9}{2}ee+\frac{15}{2}qq)\sin(2\eta-v)+\frac{3}{4m}(1+\frac{3}{2}ee+\frac{3}{4}qq) \\
\quad \sin(2\eta+v)-\frac{1}{2m}(1+\frac{3}{2}ee+\frac{1}{4}qq)\sin v \\
+\frac{3q}{4m}\sin 2\eta+\frac{3q}{2m}\sin(2\eta-2v)+\frac{3q}{4m}\sin(2\eta+2v)-\frac{q}{4m}\sin 2v \\
-\frac{27e}{8m}\sin(2\eta-v-u)-\frac{27e}{8m}\sin(2\eta-v+u)-\frac{9e}{8m}\sin(2\eta+v-u) \\
-\frac{9e}{8m}\sin(2\eta+v+u)+\frac{3e}{4m}\sin(v-u)+\frac{3e}{4m}\sin(v+u) \\
+\frac{159q}{16m}\sin(2\eta-3v)+\frac{99q}{16m}\sin(2\eta+3v)-\frac{qq}{8m}\sin 3v \\
-\frac{9eq}{8m}\sin(2\eta-u)-\frac{9eq}{8m}\sin(2\eta+u)-\frac{9eq}{4m}\sin(2\eta-2v+u) \\
-\frac{9eq}{4m}\sin(2\eta-2v-u)-\frac{9eq}{8m}\sin(2\eta+2v-u)-\frac{9eq}{8m}\sin(2\eta+2v+u) \\
+\frac{27ee}{16m}\sin(2\eta-v-2u)+\frac{27ee}{16m}\sin(2\eta-v+2u) \\
+\frac{9ee}{16m}\sin(2\eta+v-2u)+\frac{9ee}{16m}\sin(2\eta+v+2u) \\
+\frac{3eq}{8m}\sin(2v-u)+\frac{3eq}{8m}\sin(2v+u) \\
-\frac{3ee}{8m}\sin(v-2u)-\frac{3ee}{8m}\sin(v+2u) \\
+\frac{27}{8m}\xi\sin(2\eta-v)+\frac{9}{8m}\xi\sin(2\eta+v)-\frac{3}{4m}\xi\sin v
\end{cases}$$

$+\frac{9q}{8m}$

$$\begin{cases}
+\frac{9q}{8m}\xi\sin 2\eta+\frac{9q}{4m}\xi\sin(2\eta-2v)+\frac{9q}{8m}\xi\sin(2\eta+2v) \\
-\frac{3q}{8m}\xi\sin 2v+\frac{9e}{8m}\xi\sin(v-u)+\frac{9e}{8m}\xi\sin(v+u) \\
-\frac{81e}{16m}\xi\sin(2\eta-v-u)-\frac{81e}{16m}\xi\sin(2\eta-v+u) \\
-\frac{27e}{16}\xi\sin(2\eta+v-u)-\frac{27e}{16m}\xi\sin(2\eta+v+u) \\
+\frac{15n}{16m}\sin(\eta-v)-\frac{3n}{16m}\sin(\eta+v) \\
+\frac{45n}{16m}\sin(3\eta-v)+\frac{15n}{16m}\sin(3\eta+v) \\
+m\Pi\sin v-mq\Pi\sin 2v-\frac{3}{2}m\xi\Pi\sin v-i\sin v
\end{cases}$$

XVII.

Deinde cum sit $\frac{q\sin v\cos v}{1-q\cos v}=\frac{q\sin 2v}{2(1-q\cos v)}=$ $=\frac{1}{2}q\sin 2v+\frac{1}{4}qq\sin v+\frac{1}{4}qq\sin 3v$; erit pro motu elementari apogei:

$$\frac{q(d\Phi-dv)}{d\omega}=(2\sin v+\frac{1}{2}q\sin 2v+\frac{1}{4}qq\sin v$$
$$+\frac{1}{4}qq\sin 3v)\frac{M}{A}\sqrt{Ap}-\cos v.\frac{N}{A}\sqrt{Ap}$$

ac facta substitutione obtinebitur:

$\frac{q(d\Phi-dv)}{d\omega}=$

$$\frac{q(d\Phi-dv)}{d\omega}=\left\{\begin{aligned}
&+\tfrac{9}{4m}(1+\tfrac{3}{2}ee+\tfrac{7}{12}qq)\operatorname{cos}(2\eta-v)-\tfrac{3}{4m}(1+\tfrac{3}{2}ee+\tfrac{3}{4}qq)\\
&\qquad\operatorname{cos}(2\eta+v)+\tfrac{1}{2m}(1+\tfrac{3}{2}ee+\tfrac{3}{4}qq)\operatorname{cos}v\\
&+\tfrac{3q}{4m}\operatorname{cos}2\eta+\tfrac{3q}{2m}\operatorname{cos}(2\eta-2v)-\tfrac{3q}{4m}\operatorname{cos}(2\eta+2v)\\
&+\tfrac{q}{4m}\operatorname{cos}2v+\tfrac{q}{4m}-\tfrac{3e}{4m}\operatorname{cos}(v-u)-\tfrac{3e}{4m}\operatorname{cos}(v+u)\\
&-\tfrac{27e}{8m}\operatorname{cos}(2\eta-v-u)-\tfrac{27e}{8m}\operatorname{cos}(2\eta-v+u)\\
&+\tfrac{9e}{8m}\operatorname{cos}(2\eta+v-u)+\tfrac{9e}{8m}\operatorname{cos}(2\eta+v+u)\\
&+\tfrac{15qq}{16m}\operatorname{cos}(2\eta-3v)-\tfrac{9qq}{16m}\operatorname{cos}(2\eta+3v)+\tfrac{qq}{8m}\operatorname{cos}3v\\
&-\tfrac{9eq}{8m}\operatorname{cos}(2\eta-u)-\tfrac{9eq}{8m}\operatorname{cos}(2\eta+u)-\tfrac{3eq}{4m}\operatorname{cos}u\\
&-\tfrac{3eq}{8m}\operatorname{cos}(2v-u)-\tfrac{3eq}{8m}\operatorname{cos}(2v+u)\\
&-\tfrac{9eq}{4m}\operatorname{cos}(2\eta-2v+u)-\tfrac{9eq}{4m}\operatorname{cos}(2\eta-2v-u)\\
&+\tfrac{9eq}{8m}\operatorname{cos}(2\eta+2v-u)+\tfrac{9eq}{8m}\operatorname{cos}(2\eta+2v+u)\\
&+\tfrac{27ee}{16m}\operatorname{cos}(2\eta-v-2u)+\tfrac{27ee}{16m}\operatorname{cos}(2\eta-v+2u)\\
&-\tfrac{9ee}{16m}\operatorname{cos}(2\eta+v-2u)-\tfrac{9ee}{16m}\operatorname{cos}(2\eta+v+2u)\\
&+\tfrac{3ee}{8m}\operatorname{cos}(v-2u)+\tfrac{3ee}{8m}\operatorname{cos}(v+2u)
\end{aligned}\right.$$

$+\frac{27}{8m}$

$$\left|\begin{aligned}
&+\tfrac{27}{8m}\xi\operatorname{cos}(2\eta-v)-\tfrac{9}{8m}\xi\operatorname{cos}(2\eta+v)+\tfrac{3}{4m}\xi\operatorname{cos}v\\
&+\tfrac{9q}{8m}\xi\operatorname{cos}2\eta+\tfrac{9q}{4m}\xi\operatorname{cos}(2\eta-2v)-\tfrac{9q}{8m}\xi\operatorname{cos}(2\eta+2v)\\
&+\tfrac{3q}{8m}\xi+\tfrac{3q}{8m}\ \xi\operatorname{cos}2v\\
&-\tfrac{81e}{16m}\xi\operatorname{cos}(2\eta-v-u)+\tfrac{27e}{16m}\xi\operatorname{cos}(2\eta+v-u)\\
&-\tfrac{81e}{16m}\xi\operatorname{cos}(2\eta-v+u)+\tfrac{27e}{16m}\xi\operatorname{cos}(2\eta+v+u)\\
&-\tfrac{9e}{8m}\xi\operatorname{cos}(v-u)-\tfrac{9e}{8m}\xi\operatorname{cos}(v+u)\\
&+\tfrac{15n}{16m}\operatorname{cos}(\eta-v)+\tfrac{3n}{16m}\operatorname{cos}(\eta+v)\\
&+\tfrac{45n}{16m}\operatorname{cos}(3\eta-v)-\tfrac{15n}{16m}\operatorname{cos}(3\eta+v)\\
&-m\Pi\operatorname{cos}v+mq\Pi+mq\Pi\operatorname{cos}2v+\tfrac{3}{2}m\xi\Pi\operatorname{cos}v+i\operatorname{cos}v
\end{aligned}\right.$$

XVIII.

Euoluamus simili modo valorem differentialium dx et $d\varphi$, et cum sit $\frac{y}{x^3}-\frac{1}{yy}=\frac{3x}{y^3}\operatorname{cos}\eta+\frac{3xx}{4y^4}(3+5\operatorname{cos}2\eta)$ et $d\Phi=\frac{d\omega}{xx}\sqrt{Ap}$; erit

$$dx=-\frac{a^3xd\omega}{\sqrt{Ap}}\left(\frac{3x}{y^3}\operatorname{cos}\eta+\frac{3xx}{4y^4}(3+5\operatorname{cos}2\eta)\right)\operatorname{sin}(\theta-\pi)\operatorname{sin}(\Phi-\pi)$$

Substitutis autem valoribus $x=\frac{p}{1-q\operatorname{cos}v}$, $y=\frac{c}{1-e\operatorname{cos}u}$; $p=b(1+\xi)$; $\sqrt{Ap}=m\sqrt{b^3p}=mbb(1+\tfrac{1}{2}\xi)$, $\frac{a^3}{c^3}=\frac{1}{(1-ee)^3}=1+3ee$ et $\frac{b}{c}=n$, erit

$$dx=-\frac{d\omega\operatorname{sin}(\theta-\pi)\operatorname{sin}(\Phi-\pi)}{m}\left[\frac{3(1+\tfrac{3}{2}\xi)(1+3ee)(1-e\operatorname{cos}u)^3}{(1-q\operatorname{cos}v)^2}\operatorname{cos}\eta+\tfrac{1}{4}n(3+5\operatorname{cos}2\eta)\right]$$

 Negle-

+⅞qq) +2v) v+u) u)))ſ3v u Euolu et cum f dx=-: Subſtitu √Ap=n dx=-du 27/m

Neglectis igitur terminis, qui nullum valorem sensibilem continent, habebimus

$$\frac{d\pi}{d\omega} = \left\{ \begin{array}{l} -\frac{3}{4m}(1+\frac{9}{2}ee+\frac{3}{2}qq) - \frac{3}{4m}(1+\frac{9}{2}ee+\frac{3}{2}qq)\cos 2\eta \\ +\frac{3}{4m}(1+\frac{9}{2}ee+\frac{3}{2}qq)\cos 2(\Phi-\pi) + \frac{3}{4m}(1+\frac{9}{2}ee+\frac{3}{2}qq)\cos 2(\theta-\pi) \\ -\frac{3q}{2m}\cos v - \frac{3q}{4m}\cos(2\eta-v) - \frac{3q}{4m}\cos(2\eta+v) \\ +\frac{9e}{4m}\cos u + \frac{9e}{8m}\cos(2\eta-u) + \frac{9e}{8m}\cos(2\eta+u). \\ +\frac{3q}{4m}\cos(2\Phi-2\pi-v) + \frac{3q}{4m}\cos(2\Phi-2\pi+v) \\ +\frac{3q}{4m}\cos(2\theta-2\pi-v) + \frac{3q}{4m}\cos(2\theta-2\pi+v) \\ -\frac{9e}{8m}\cos(2\Phi-2\pi-u) - \frac{9e}{8m}\cos(2\Phi-2\pi+u) \\ -\frac{9e}{8m}\cos(2\theta-2\pi-u) - \frac{9e}{8m}\cos(2\theta-2\pi+u) \\ -\frac{9}{8m}\xi - \frac{9}{8m}\xi\cos 2\eta \\ +\frac{9}{8m}\xi\cos(2\Phi-2\pi) + \frac{9}{8m}\xi\cos(2\theta-2\pi) \\ -\frac{11n}{16m}\cos\eta + \frac{3n}{8m}\cos(\Phi+\theta-2\pi) - \frac{5n}{16m}\cos 3\eta \\ +\frac{5n}{16m}\cos(3\Phi-\theta-2\pi) + \frac{5n}{16m}\cos(3\theta-\Phi-2\pi) \end{array} \right.$$

XIX.

Sim ... sibi-
tang (Φ·
tangentis ... ſ 2η
-π)
v)
u).
d.ltang ϱ / dω

XIX.

Simili autem modo praecedentem valorem per tang $(\Phi-\pi)$ diuidendo prodibit differentiale logarithmi tangentis inclinationis ϱ, erit enim

$$\frac{d.l\,\mathrm{tang}\,\varrho}{d\omega} = \left\{ \begin{array}{l} +\frac{3}{4m}(1+\frac{9}{2}ee+\frac{3}{2}qq)\sin 2\eta - \frac{3}{4m}(1+\frac{9}{2}ee+\frac{3}{2}qq) \\ \qquad \sin 2(\Phi-\pi) - \frac{3}{4m}(1+\frac{9}{2}ee+\frac{3}{2}qq)\sin 2(\theta-\pi) \\ +\frac{3q}{4m}\sin(2\eta-v) + \frac{3q}{4m}\sin(2\eta+v) \\ -\frac{9e}{8m}\sin(2\eta-u) - \frac{9e}{8m}\sin(2\eta+u) \\ -\frac{3q}{4m}\sin(2\Phi-2\pi-v) - \frac{3q}{4m}\sin(2\Phi-2\pi+v) \\ -\frac{3q}{4m}\sin(2\theta-2\pi-v) - \frac{3q}{4m}\sin(2\theta-2\pi+v) \\ +\frac{9e}{8m}\sin(2\Phi-2\pi-u) + \frac{9e}{8m}\sin(2\Phi-2\pi+u) \\ +\frac{9e}{8m}\sin(2\theta-2\pi-u) + \frac{9e}{8m}\sin(2\theta-2\pi+u) \\ +\frac{9}{8m}\xi\sin 2\eta - \frac{9}{8m}\xi\sin(2\Phi-2\pi) - \frac{9}{8m}\xi\sin(2\theta-2\pi) \\ +\frac{n}{16m}\sin\eta - \frac{3n}{8m}\sin(\Phi+\theta-\pi) + \frac{5n}{16m}\sin 3\eta \\ -\frac{5n}{16m}\sin(3\Phi-\theta-2\pi) - \frac{5n}{16m}\sin(3\theta-\Phi-2\pi) \end{array} \right.$$

XX.

XX.

Quo iam facilius has formulas admodum complicatas euoluere queamus, quadruplicis generis terminos diſtingui conuenit. Primum ſcilicet genus eos complectitur terminos, qui tantum ab excentricitate orbitae lunaris pendent, neque excentricitatem ſolis, neque parallaxin ſolis ſeu literam π, neque inclinationem orbitae lunaris ſeu litteram Π inuoluunt. Ad ſecundum genus refero terminos, qui ad primum genus inſuper excentricitatem ſolis adiungunt. Ad tertium autem eos, qui praeterea parallaxin ſolis ſeu litteram π inducunt. In quarto autem eas inaequalitates, quae inſuper ab obliquitate orbitae lunaris proueniunt, complexurus ſum. Ab inaequalitatibus ergo primi generis exordiar, ideoque cum excentricitatem ſolis e, tum eius parallaxin, tum quoque obliquitatem orbitae lunaris reiiciam

INUESTIGATIO INAEQUALITATUM
LUNAE PRIMI GENERIS.

XXI.

Neglectis ergo excentricitate ſolis cum eius parallaxi et obliquitate orbitae lunaris, has habebimus aequationes:

$\frac{d\xi}{d\omega} =$

$$\frac{d\xi}{d\omega} = \begin{cases} -\frac{3}{m}(1+\frac{3}{2}qq)\,\text{ſin}\,2\eta - \frac{3q}{m}\,\text{ſin}\,(2\eta-v) - \frac{3q}{m}\,\text{ſin}\,(2\eta+v) \\ -\frac{9qq}{4m}\,\text{ſin}\,(2\eta-2v) - \frac{9qq}{4m}\,\text{ſin}\,(2\eta+2v) \\ -\frac{15}{2m}\xi\,\text{ſin}\,2\eta - \frac{15q}{2m}\xi\,\text{ſin}\,(2\eta-v) - \frac{15q}{2m}\xi\,\text{ſin}\,(2\eta+v) \end{cases}$$

$$\frac{dq}{d\omega} = \begin{cases} +\frac{9}{4m}(1+\frac{5}{12}qq)\,\text{ſin}\,(2\eta-v) + \frac{3}{4m}(1+\frac{3}{4}qq)\,\text{ſin}\,(2\eta+v) \\ -\frac{1}{2m}(1+\frac{1}{4}qq)\,\text{ſin}\,v - i\,\text{ſin}\,v \\ +\frac{3q}{4m}\,\text{ſin}\,2\eta + \frac{3q}{2m}\,\text{ſin}\,(2\eta-2v) + \frac{3q}{4m}\,\text{ſin}\,(2\eta+2v) \\ -\frac{q}{4m}\,\text{ſin}\,2v \\ +\frac{15qq}{16m}\,\text{ſin}\,(2\eta-3v) + \frac{9qq}{16m}\,\text{ſin}\,(2\eta+3v) - \frac{qq}{8m}\,\text{ſin}\,3v \\ +\frac{27}{8m}\xi\,\text{ſin}\,(2\eta-v) + \frac{9}{8m}\xi\,\text{ſin}\,(2\eta+v) - \frac{3}{4m}\xi\,\text{ſin}\,v \\ +\frac{9q}{8m}\xi\,\text{ſin}\,2\eta + \frac{9q}{4m}\xi\,\text{ſin}\,(2\eta-2v) + \frac{9q}{8m}\xi\,\text{ſin}\,(2\eta+2v) \\ -\frac{3q}{8m}\xi\,\text{ſin}\,2v \end{cases}$$

$\frac{q(d\Phi\ dv)}{d\omega} =$

$$\frac{q(d\Phi - dv)}{d\omega} = \begin{cases} + \frac{9}{4m}(1+\frac{7}{12}qq)\cos(2\eta - v) - \frac{3}{4m}(1+\frac{1}{4}qq)\cos(2\eta + v) \\ + \frac{1}{2m}(1+\frac{3}{4}qq)\cos v + i \cos v \\ + \frac{3q}{4m}\cos 2\eta + \frac{3q}{2m}\cos(2\eta - 2v) - \frac{3q}{4m}\cos(2\eta + 2v) \\ + \frac{q}{4m}\cos 2v + \frac{q}{4m} \\ + \frac{15qq}{16m}\cos(2\eta - 3v) - \frac{9qq}{16m}\cos(2\eta + 3v) + \frac{qq}{8m}\cos 3v \\ + \frac{27}{8m}\xi\cos(2\eta - v) - \frac{9}{8m}\xi\cos(2\eta + v) + \frac{3}{4m}\xi\cos v \\ + \frac{9q}{8m}\xi\cos 2\eta + \frac{9q}{4m}\xi\cos(2\eta - 2v) - \frac{9q}{8m}\xi\cos(2\eta + 2v) \\ + \frac{3q}{8m}\xi + \frac{3q}{8m}\xi\cos 2v \end{cases}$$

$$\frac{d\Phi}{d\omega} = m(1+\tfrac{1}{2}qq) - 2mq\cos v + \tfrac{1}{2}mqq\cos 2v - \tfrac{3}{2}m(1+\tfrac{1}{2}qq)\xi$$
$$+ 3mq\xi\cos v - \tfrac{3}{4}mqq\xi\cos 2v + \tfrac{15}{8}m\xi\xi$$

$$\frac{d\eta}{d\omega} = m(1+\tfrac{1}{2}qq) - 1 - 2mq\cos v + \tfrac{1}{2}mqq\cos 2v - \tfrac{3}{2}m(1+\tfrac{1}{2}qq)\xi$$
$$+ 3mq\xi\cos v - \tfrac{3}{4}mqq\xi\cos 2v + \tfrac{15}{8}m\xi\xi$$

XXII.

Hic autem primo patet valores litterarum ξ et q fine cognitis rationibus $\frac{d\eta}{d\omega}$ et $\frac{dv}{d\omega}$ definiri non posse, has autem vicissim ipsas quantitates ξ et q inuoluere. Cum autem ad valores ξ et q inueniendos non opus sit rationes $\frac{d\eta}{d\omega}$ et $\frac{dv}{d\omega}$ eo praecisionis gradu nosse, quo ipsi illi valores desiderantur; patet si valores ξ et q prope tantum veri constent, iis in rationibus $\frac{d\eta}{d\omega}$ et $\frac{dv}{d\omega}$ adhibitis, eosdem multo exactiores repertum iri. Cum igitur, si motus esset regularis, foret $\xi = o$ et $q =$ constanti, hinc primam hypothesin constituamus. Sit ergo

$$\xi = o \quad \text{et} \quad q = g$$

et neglectis terminis, qui ob harum litterarum errores affici possent, vtpote valde paruis prae reliquis, habebimus proxime

$$\frac{d\Phi}{d\omega} = m(1+\tfrac{1}{2}gg) - 2mg\cos v; \quad \frac{d\eta}{d\omega} = (1+\tfrac{1}{2}gg) - 1 - 2mg\cos v$$

et
$$\frac{d\Phi - dv}{d\omega} = \frac{9}{4mg}(1+\tfrac{7}{12}gg)\cos(2\eta - v) - \frac{3}{4mg}(1+\tfrac{1}{4}gg)\cos(2\eta + v)$$
$$+ \frac{1}{2mg}(1+\tfrac{3}{4}gg)\cos v + \frac{i}{g}\cos v + \frac{1}{4m}$$

ideoque

$$\frac{dv}{d\omega} = m(1+\tfrac{1}{2}gg) - \frac{1}{4m} - \left(2mg + \frac{1}{2mg} + \frac{3g}{8m} + \frac{i}{g}\right)\cos v$$
$$- \frac{9}{4mg}(1+\tfrac{7}{12}gg)\cos(2\eta - v) + \frac{3}{4mg}(1+\tfrac{1}{4}gg)\cos(2\eta + v)$$

XXIII.

Ponamus ad has formulas abbreuiandas:

$$m(1+\tfrac{1}{2}gg) - 1 = \alpha; \qquad 2mg = \gamma$$
$$m(1+\tfrac{1}{2}gg) - \frac{1}{4m} = \beta; \quad 2mg + \frac{1}{2mg} + \frac{3g}{8m} + \frac{i}{g} = \delta$$

 et

et neglectis quadratis gg in reliquis terminis, habebimus has formulas simpliciores:

$$\frac{d\eta}{d\omega} = \alpha - \gamma \operatorname{cof} v$$

$$\frac{dv}{d\omega} = \mathcal{E} - \delta \operatorname{cof} v - \frac{9}{4mg} \operatorname{cof}(2\eta - v) + \frac{3}{4mg} \operatorname{cof}(2\eta + v)$$

Tum vero pro valoribus ξ et q propius inueniendis has aequationes:

$$\frac{d\xi}{d\omega} = -\frac{3}{m}(1+\tfrac{3}{4}gg)\operatorname{fin} 2\eta - \frac{3g}{m}\operatorname{fin}(2\eta - v) - \frac{3g}{m}\operatorname{fin}(2\eta + v)$$

$$\frac{dq}{d\omega} = + \frac{9}{4m}(1+\tfrac{1}{12}gg)\operatorname{fin}(2\eta - v) + \frac{3}{4m}(1+\tfrac{3}{4}gg)\operatorname{fin}(2\eta + v)$$

$$- \frac{1}{2m}(1+\tfrac{3}{4}gg)\operatorname{fin} v - i \operatorname{fin} v$$

$$+ \frac{3g}{4m}\operatorname{fin} 2\eta + \frac{3g}{2m}\operatorname{fin}(2\eta - 2v) + \frac{3g}{4m}\operatorname{fin}(2\eta + 2v) - \frac{g}{4m}\operatorname{fin} 2v$$

XXIV.

Fingamus ergo primo:

$$\xi = \mathfrak{A} \operatorname{cof} 2\eta + \mathfrak{B} \operatorname{cof}(2\eta - v) + \mathfrak{C} \operatorname{cof}(2\eta + v)$$

vbi notandum est terminos binos posteriores, vti in differentiali, multo esse minores primo. Quare cum etiam in differentialibus $d\eta$ et dv duplicis generis termini occurrant, quorum posteriores prae primis sint valde parui, in differentiatione solius primi termini totum differentialis $d\eta$ valorem pono, in duobus vero reliquis tantum valorem principalem; sic prodibit

$$\frac{d\xi}{d\omega} = -2\alpha\mathfrak{A}\operatorname{fin} 2\eta + \gamma\mathfrak{A}\operatorname{fin}(2\eta - v) + \gamma\mathfrak{A}\operatorname{fin}(2\eta + v)$$

$$-(2\alpha - \mathcal{E})\mathfrak{B} \qquad -(2\alpha + \mathcal{E})\mathfrak{C}$$

Collato ergo hoc differentiali cum forma proposita obtinetur:

$$\mathfrak{A} = \frac{3}{2m\alpha}(1+\tfrac{3}{4}gg)$$

$$(2\alpha - \mathcal{E})\mathfrak{B} = \gamma\mathfrak{A} + \frac{3g}{m} \quad \text{ergo} \quad \mathfrak{B} = \frac{3(\gamma + 2\alpha g)}{2m\alpha(2\alpha - \mathcal{E})}$$

$$(2\alpha + \mathcal{E})\mathfrak{C} = \gamma\mathfrak{A} + \frac{3g}{m} \quad \text{ergo} \quad \mathfrak{C} = \frac{3(\gamma + 2\alpha g)}{2m\alpha(2\alpha + \mathcal{E})}$$

XXV.

Simili modo fingatur:

$$q = g + A\operatorname{cof}(2\eta - v) + B\operatorname{cof}(2\eta + v) + C\operatorname{cof} v$$

$$+ D\operatorname{cof} 2\eta + E\operatorname{cof}(2\eta - 2v) + F\operatorname{cof}(2\eta + 2v) + G\operatorname{cof} 2v$$

$$+ H\operatorname{cof} 4\eta + J\operatorname{cof}(4\eta - 2v) + K\operatorname{cof}(4\eta + 2v)$$

vbi linea prior continet terminos multo maiores, quam binae inferiores. Hinc ergo fit differentiando secundum regulam supra datam:

$$\frac{dq}{d\omega} = -(2\alpha - \mathcal{E})A\operatorname{fin}(2\eta - v) - (2\alpha + \mathcal{E})B\operatorname{fin}(2\eta + v) - \mathcal{E}C\operatorname{fin} v$$

$$+ \left(\tfrac{1}{2}(2\gamma - \delta)A + \tfrac{1}{2}(2\gamma + \delta)B + \frac{3}{2mg}C - 2\alpha D\right)\operatorname{fin} 2\eta$$

$$+ \left(\tfrac{1}{2}(2\gamma - \delta)A - \frac{9}{8mg}C - 2(\alpha - \mathcal{E})E\right)\operatorname{fin}(2\eta - 2v)$$

$$+ \left(\tfrac{1}{2}(2\gamma + \delta)B - \frac{3}{8mg}C - 2(\alpha + \mathcal{E})F\right)\operatorname{fin}(2\eta + 2v)$$

$$+ \left(\tfrac{1}{2}\delta C - \frac{3A + 9B}{8mg} - 2\mathcal{E}G\right)\operatorname{fin} 2v$$

$$+ \left(\frac{3A + 9B}{8mg} - 4\alpha H\right)\operatorname{fin} 4\eta$$

$$+ \left(-\frac{9A}{8mg} - 2(2\alpha - \mathcal{E})J\right)\operatorname{fin}(4\eta - 2v)$$

$$+ \left(-\frac{3B}{8mg} - 2(2\alpha + \mathcal{E})K\right)\operatorname{fin}(4\eta + 2v)$$

Hincque elicientur sequentes coefficientium valores:

$$(2\alpha-\beta)\,A = -\frac{9}{4m}\left(1+\tfrac{5}{12}gg\right)$$

$$(2\alpha+\beta)\,B = -\frac{3}{4m}\left(1+\tfrac{3}{4}gg\right)$$

$$\beta C = \frac{1}{2m}\left(1+\tfrac{1}{4}gg\right)+i$$

$$2\alpha D = \tfrac{1}{2}(2\gamma-\delta)A+\tfrac{1}{2}(2\gamma+\delta)B+\frac{3}{2mg}C-\frac{3g}{4m}$$

$$2(\alpha-\beta)E = \tfrac{1}{2}(2\gamma-\delta)A-\frac{9}{8mg}C-\frac{3g}{2m}$$

$$2(\alpha+\beta)F = \tfrac{1}{2}(2\gamma+\delta)A-\frac{3}{8mg}C-\frac{3g}{4m}$$

$$2\beta G = -\frac{3A+9B}{8mg}+\tfrac{1}{2}\delta C+\frac{g}{4m}$$

$$4\alpha H = \frac{3A+9\beta}{8mg};\quad 2(2\alpha-\beta)J = -\frac{9}{8mg}A$$

$$2(2\alpha+\beta)K = -\frac{3}{8mg}B$$

XXVI.

Cum igitur his inuentis valoribus fit multo verius:

$$\xi = \mathfrak{A}\cos 2\eta \text{ et } q = g + A\cos(2\eta-v)+B\cos(2\eta+v)+C\cos v$$

vbi terminos minores data opera adhuc omitto, quia fortasse correctione egent, praecedentes operationes multo accuratius instituere atque ad ordinem terminorum vlteriorem progredi poterimus. Obtinebimus ergo:

$$\frac{d\Phi}{d\omega} = m\left(1+\tfrac{1}{2}gg-C\right) - 2mg\cos v$$
$$-m\left(\tfrac{3}{2}\mathfrak{A}+A+B\right)\cos 2\eta - mA\cos(2\eta-2v) - mB\cos(2\eta+2v) + m\left(\tfrac{1}{2}gg-C\right)\cos 2v$$

hincque $\frac{d\eta}{d\omega} = \frac{d\Phi}{d\omega} - 1$.

Porro

Porro ob $\frac{1}{q} = \frac{1}{g} - \frac{A}{gg}\cos(2\eta-v) - \frac{B}{gg}\cos(2\eta+v) - \frac{C}{gg}\cos v$ erit

$$\frac{d\Phi-dv}{d\omega} = \frac{9}{4mg}\left(1+\tfrac{7}{12}gg\right)\cos(2\eta-v) - \frac{3}{4mg}\left(1+\tfrac{1}{4}gg\right)\cos(2\eta+v)$$
$$+\frac{1}{2mg}\left(1+\tfrac{3}{4}gg\right)\cos v + \frac{i}{g}\cos v + \frac{1}{4m}\left(1-\frac{9A+3B-2C}{2gg}\right)$$
$$+\frac{1}{4m}\left(3-\frac{A-B-3C}{gg}\right)\cos 2\eta + \frac{1}{4m}\left(6-\frac{2A-9C}{2gg}\right)\cos(2\eta-2v)$$
$$-\frac{1}{4m}\left(3+\frac{2B-3C}{2gg}\right)\cos(2\eta+2v) + \frac{1}{4m}\left(1+\frac{3A-9B-2C}{2gg}\right)\cos 2v$$
$$+\frac{3A-9B}{8mgg}\cos 4\eta - \frac{9A}{8mgg}\cos(4\eta-2v) + \frac{3B}{8mgg}\cos(4\eta+2v)$$

XXVII.

Ponatur ad abbreuiandum:

$$m\left(1+\tfrac{1}{2}gg-C\right)-1 = \alpha;\quad 2mg = \gamma$$
$$m\left(\tfrac{3}{2}\mathfrak{A}+A+B\right) = \epsilon;\quad \text{vt fit}$$

$$\frac{d\eta}{d\omega} = \alpha - \gamma\cos v - \epsilon\cos 2\eta$$
$$-mA\cos(2\eta-2v) - mB\cos(2\eta+2v) + m\left(\tfrac{1}{2}gg-C\right)\cos 2v$$

Porro fit

$$m\left(1+\tfrac{1}{2}gg-C\right) - \frac{1}{4m}\left(1-\frac{9A+3B-2C}{2gg}\right) = \beta$$

$$2mg + \frac{1}{2mg} + \frac{3g}{8m} + \frac{i}{g} = \delta$$

$$+m\left(\tfrac{3}{2}\mathfrak{A}+A+B\right) + \frac{1}{4m}\left(3-\frac{A-B-3C}{gg}\right) = \zeta$$

mA

$$mA + \frac{1}{4m}\left(6 - \frac{2A-9C}{2gg}\right) = \eta$$

$$mB - \frac{1}{4m}\left(3 + \frac{2B-3C}{2gg}\right) = \theta$$

$$m(C - \tfrac{1}{2}gg) + \frac{1}{4m}\left(1 + \frac{3A-9B-2C}{2gg}\right) = \varkappa$$

vt habeatur

$$\frac{dv}{d\omega} = \epsilon - \delta \cos v - \frac{9}{4mg}\cos(2\eta - v) + \frac{3}{4mg}\cos(2\eta + v)$$
$$- \zeta \cos 2\eta - \eta \cos(2\eta - 2v) - \theta \cos(2\eta + 2v) - \varkappa \cos 2v$$
$$- \frac{3A+9B}{8mgg}\cos 4\eta + \frac{9A}{8mgg}\cos(4\eta - 2v) - \frac{3B}{8mgg}\cos(4\eta + 2v)$$

vbi caueatur, ne coefficientes η, θ, cum angulis cognominibus confundantur.

XXVIII.

Opus plane non eſt, vt valores litterarum ξ et q accuratius determinemus, atque ad plures terminos, quam ante inuenimus, expediamus; verum hos ipſos terminos, quos ante inuenimus, accuratius obtinebimus, ſi litteris α et ϵ eos valores tribuemus, quos nunc eis conuenire collegimus. Pluribus autem terminis non indigebimus tam ad longitudinem lunae Φ, quam ad eius anomaliam veram v ſatis exacte definiendam. Verum ad hoc ipſum negotium valores differentiales $\frac{d\Phi}{d\omega}$ et $\frac{d\Phi - dv}{d\omega}$, ac praecipue hunc poſteriorem, quo motus apogei continetur, accuratius euolui oportet, quoniam imprimis in motu medio apogei minimae particulae ingentis momenti eſſe poſſunt.

XXIX.

XXIX.

Cum igitur accuratius quam adhuc aſſumſimus ſit

$$\xi = \mathfrak{A}\cos 2\eta + \mathfrak{B}\cos(2\eta - v) + \mathfrak{C}\cos(2\eta + v) \quad \text{et}$$

$$q = g + A\cos(2\eta - v) + B\cos(2\eta + v) + C\cos v$$
$$+ D\cos 2\eta + E\cos(2\eta - 2v) + F\cos(2\eta + 2v) + G\cos 2v$$
$$+ H\cos 4\eta + J\cos(4\eta - 2v) + K\cos(4\eta + 2v)$$

erit terminus ad quartum vsque ordinem extenſis

I. II.

$$\frac{d\Phi}{d\omega} = m(1 + \tfrac{1}{2}gg - C) - (2mg - \tfrac{3}{2}mgC + mG)\cos v$$

III.

$$-m(\tfrac{3}{2}\mathfrak{A} + A + B)\cos 2\eta - mA\cos(2\eta - 2v) - mB\cos(2\eta + 2v)$$
$$- m(C - \tfrac{1}{2}gg)\cos 2v$$

IV.

$$+ m(\tfrac{3}{2}g\mathfrak{A} - \tfrac{3}{2}\mathfrak{B} + gA + \tfrac{1}{2}gB - D - E)\cos(2\eta - v)$$
$$+ m(\tfrac{3}{2}g\mathfrak{A} - \tfrac{3}{2}\mathfrak{C} + gB + \tfrac{1}{2}gA - D - F)\cos(2\eta + v)$$
$$+ m(\tfrac{1}{2}gA - E)\cos(2\eta - 3v) + m(\tfrac{1}{2}gB - F)\cos(2\eta + 3v)$$
$$+ m(\tfrac{1}{2}gC - G)\cos 3v$$
$$- m(H + J)\cos(4\eta - v) - m(H + K)\cos(4\eta + v) - mJ\cos(4\eta - 3v)$$
$$- mK\cos(4\eta + 3v)$$

vnde cum eſſet ante $\gamma = 2mg$, nunc accuratius erit

$$\gamma = 2mg - \tfrac{3}{2}mgC + mG$$

XXX.

XXX.

Deinde cum nunc quoque ſit accuratius;

$$\frac{1}{q} = \left(\frac{1}{g} + \overset{\text{II.}}{\frac{AA + BB + CC}{2g^3}}\right)$$

III.

$$-\frac{A}{gg}\operatorname{cof}(2\eta - v) - \frac{B}{gg}\operatorname{cof}(2\eta + v) - \frac{C}{gg}\operatorname{cof} v$$

IV.

$$+\left(\frac{(A+B)C}{g^3} - \frac{D}{gg}\right)\operatorname{cof} 2\eta + \left(\frac{AC}{g^3} - \frac{E}{gg}\right)\operatorname{cof}(2\eta - 2v)$$

$$+\left(\frac{BC}{g^3} - \frac{F}{gg}\right)\operatorname{cof}(2\eta + 2v) + \left(\frac{2AB + CC}{2g^3} - \frac{G}{gg}\right)\operatorname{cof} 2v$$

$$+\left(\frac{AB}{g^3} - \frac{H}{gg}\right)\operatorname{cof} 4\eta + \left(\frac{AA}{2g^3} - \frac{J}{gg}\right)\operatorname{cof}(4\eta - 2v)$$

$$+\left(\frac{BB}{g2^3} - \frac{K}{gg}\right)\operatorname{cof}(4\eta + 2v)$$

Hinc quoque ad terminos quarti ordinis vsque valor formulae $\frac{d\Phi - dv}{d\omega}$ definiri poſſet, ſed expreſſio prodiret tantopere complicata, vt eius euolutio ſummam requireret patientiam; neque tamen hic labor vllius foret vſus, niſi forte in motu apogei exactius eruendo: ipſae enim inaequalitates nullius forent momenti; propterea quod error in anomalia commiſſus multo minorem errorem in longitudine producit.

XXXI.

XXXI.

Ponatur ergo longitudo apogei:

$$\Phi - v = \text{Conſt.}$$

$$+ A' \operatorname{fin}(2\eta - v) + B' \operatorname{fin}(2\eta + v) + C' \operatorname{fin} 2v$$

$$+ \Delta\omega + D' \operatorname{fin} 2\eta + E' \operatorname{fin}(2\eta - 2v) + F' \operatorname{fin}(2\eta + 2v) + G' \operatorname{fin} 2v$$

$$+ H' \operatorname{fin} 4\eta + J \operatorname{fin}(4\eta - 2v) + K \operatorname{fin}(4\eta + 2v)$$

et erit differentiando:

$$\frac{d\Phi - dv}{d\omega} =$$

$$(2\alpha - \beta) A' \operatorname{cof}(2\eta - v) + (2\alpha + \beta) B' \operatorname{cof}(2\eta + v) + \beta C' \operatorname{cof} v$$

$$+ \Delta - \tfrac{1}{2}\delta C' + \frac{9A'}{8mg} + \frac{3B'}{8mg} - \epsilon D' - (mA - \eta) E'$$

$$- (mB + \theta) F' - \kappa G' - \frac{9AJ'}{8mgg} - \frac{3BK'}{3mgg}$$

$$\operatorname{cof} 2\eta \left(-\tfrac{1}{2}(2\gamma - \delta) A' - \tfrac{1}{2}(2\gamma + \delta) B' - \frac{3C'}{4mg} + 2\alpha D'\right.$$

$$\operatorname{cof}(2\eta - 2v)\left(-\tfrac{1}{2}(2\gamma - \delta) A' - \frac{9C'}{8mg} + 2(\alpha - \beta) E'\right.$$

$$\operatorname{cof}(2\eta + 2v)\left(-\tfrac{1}{2}(2\gamma + \delta) B' + \frac{3C'}{8mg} + 2(\alpha + \beta) F'\right.$$

$$\operatorname{cof} 2v \left(-\tfrac{1}{2}\delta C' - \frac{3A'}{8mg} - \frac{9B'}{8mg} + 2\beta G'\right.$$

$$\operatorname{cof} 4\eta \left(-\frac{3A'}{8mg} - \frac{9B'}{8mg} + 4\alpha H'\right.$$

$$\operatorname{cof}(4\eta - 2v)\left(\frac{9A'}{8mg} + 2(2\alpha - \beta) J'\right.$$

$$\operatorname{cof}(4\eta + 2v)\left(\frac{3B'}{8mg} + 2(2\alpha + \beta) K'\right.$$

XXXII.

XXXIII.

Calculo autem praecipue in primis terminis accuratius expedito eſt :

$$\frac{d\Phi - dv}{d\omega} =$$

$$+\frac{1}{4mg}\operatorname{cof}(2\eta-v)\left(9+\tfrac{21}{4}gg+\frac{27AA+18BB+15CC}{4gg}\right.$$
$$\left.-\frac{3AB+2AC+BC}{gg}-\frac{2D-2E+3G+3H-9J}{2g}\right)$$

$$+\frac{1}{4mg}\operatorname{cof}(2\eta+v)\left(-3-\tfrac{3}{4}gg-\frac{6AA-9BB+15CC}{4gg}\right.$$
$$\left.+\frac{9AB+AC+2BC}{gg}-\frac{2D-2F-9G-9H+3K}{2g}\right)$$

$$+\frac{1}{2mg}\operatorname{cof}v\left(1+\tfrac{3}{4}gg+\frac{2AA+2BB+3CC}{4gg}\right.$$
$$\left.+\frac{AB+6AC+6BC}{2gg}-\frac{3D-3E-3F-G}{2g}+2mi\right)$$

$$+\frac{1}{4m}\left(1-\frac{9A+3B-2C}{2gg}\right)$$

$$+\frac{1}{4m}\left(3-\frac{A-B-3C}{gg}\right)\operatorname{cof}2\eta+\frac{1}{4m}\left(1+\frac{3A-9B-2C}{2gg}\right)\operatorname{cof}2v$$

$$+\frac{1}{4m}\left(6-\frac{2A-9C}{2gg}\right)\operatorname{cf}(2\eta-2v)-\frac{1}{4m}\left(3+\frac{2B-3C}{2gg}\right)\operatorname{cf}(2\eta+2v)$$

$$+\frac{3A-9B}{8mgg}\operatorname{cof}4\eta-\frac{9A}{8mgg}\operatorname{cof}(4\eta-2v)+\frac{3B}{8mgg}\operatorname{cof}(4\eta+2v)$$

Simili

Simili autem modo ex valore ipſius $\frac{dq}{d\omega}$ accuratius erit

$$(2\alpha-\beta)A = -\frac{1}{4m}\left(9+\tfrac{15}{4}gg+\tfrac{9}{2}C\right)$$

$$(2\alpha+\beta)B = -\frac{1}{4m}\left(3+\tfrac{9}{4}gg+3C\right)$$

$$\beta C = \frac{1}{2m}\left(1+\tfrac{1}{4}gg+\tfrac{3}{4}A+2mi\right)$$

XXXIII.

Comparatione autem inſtituta reperitur :

$$(2\alpha-\beta)A' = \frac{1}{4mg}\left(9+\tfrac{21}{4}gg+\frac{27AA+18BB+15CC}{4gg}\right.$$
$$\left.-\frac{3AB+2AC+BC}{gg}-\frac{2D-2E+3G+3H-9J}{2g}\right)$$

ſeu

$$(2\alpha-\beta)A' = -\frac{1}{g}(2\alpha-\beta)A+\frac{1}{4mg}\left(\tfrac{3}{2}gg-\tfrac{9}{2}C+\frac{27AA+18BB+15CC}{4gg}\right.$$
$$\left.-\frac{3AB+2AC+BC}{gg}-\frac{2D-2E+3G+3H-9J}{2g}\right)$$

$$(2\alpha+\beta)B' = +\frac{1}{g}(2\alpha+\beta)B+\frac{1}{4mg}\left(\tfrac{3}{2}gg+3C-\frac{6AA-9BB+15CC}{4gg}\right.$$
$$\left.+\frac{9AB+AC+2BC}{gg}-\frac{2D-2F-9G-9H+3K}{2g}\right)$$

$$\beta C' = +\frac{1}{g}\beta C+\frac{1}{2mg}\left(\tfrac{1}{2}gg-\tfrac{3}{4}A+\frac{2AA+2BB+3CC}{4gg}\right.$$
$$\left.+\frac{AB+6AC+6BC}{2gg}-\frac{3D-3E-3F-G}{2g}\right)$$

Quibus

Quibus valoribus ſubſtitutis obtinebitur pro apogei motu medio, qui in termino $\Delta\omega$ continetur:

$$\Delta = \frac{1}{4m} + \frac{2mg\delta - 1}{4mgg}C$$
$$+ \frac{\delta}{4\beta mg}\Big(\tfrac{1}{2}gg - \tfrac{3}{4}A + \frac{2AA + 2BB + 3CC}{4gg}$$
$$+ \frac{AB + 6AC + 6BC}{2gg} - \frac{3D - 3E - 3F - G}{2g}\Big)$$
$$- \frac{9}{32(2\alpha-\beta)mmgg}\Big(\tfrac{3}{2}gg - \tfrac{9}{4}C + \frac{27AA + 18BB + 15CC}{4gg}$$
$$- \frac{3AB + 2AC + BC}{gg} - \frac{2D - 2E + 3G + 3H - 9J}{2g}\Big)$$
$$- \frac{3}{32(2\alpha+\beta)mmgg}\Big(\tfrac{1}{2}gg + 3C - \frac{6AA - 9BB + 15CC}{4gg}$$
$$+ \frac{9AB + AC + 2BC}{gg} - \frac{2D - 2F - 9G - 9H + 3K}{2g}\Big)$$
$$+ \varepsilon D' + (mA - \eta)E' + (mB + \theta)F' + \varkappa G' + \frac{9AJ'}{8mgg} + \frac{3BK'}{8mgg}$$

Quae expreſſio, cum omnino ſit ſimilis illi, quae methodo praecedente eſt inuenta, nullum etiam dubium relinquit, quin et hinc motus apogei proditurus ſit obſeruationibus conformis; ideoque littera illa ε omitti poterit.

XXXIV.

Hinc igitur patet ad motum apogei definiendum valores litterarum A, B, C et A′, B′, C′ ſumma accuratione inueſtigari debere, qui cum conſtent partibus duplicis ordinis, etiam ſi partes poſterioris ordinis prae primo

primo admodum videantur paruae, eas tamen omni cura euolui oportet, propterea quod pro motu apogei partes primi ordinis ſe deſtruunt. Quod cum in determinatione reliquorum coefficientium vſu non eueniat, in his quoque non erit opus, vt partes iſtae minores in computum ducantur, ſed ſufficiet partibus principalibus vti. Scilicet etſi determinatio litterae Δ maxime eſt lubrica, atque a reliquorum coefficientium exactiſſimis valoribus pendet, reliqui tamen coefficientes tantam ſollertiam minime requirunt, ſed ſatis exacte ſine tanta opera definiri poſſunt.

XXXV.

Valores ergo reliquorum coefficientium ſequenti modo neglectis exiguis particulis ita ſe habebunt,

$$(2\alpha-\beta)A' = \frac{9}{4mg}; \quad (2\alpha+\beta)B' = -\frac{3}{4mg}; \quad \beta C' = \frac{1}{2mg}.$$
$$2\alpha D' = \tfrac{1}{2}(2\gamma-\delta)A' + \tfrac{1}{2}(2\gamma+\delta)B' + \frac{3C'}{4mg} + \frac{1}{4m}\Big(3 - \frac{A - B - 3C}{gg}\Big)$$
$$2(\alpha-\beta)E' = \tfrac{1}{2}(2\gamma-\delta)A' + \frac{9C'}{8mg} + \frac{1}{4m}\Big(6 - \frac{2A - 9C}{2gg}\Big)$$
$$2(\alpha+\beta)F' = \tfrac{1}{2}(2\gamma+\delta)B' - \frac{3C'}{8mg} - \frac{1}{4m}\Big(3 + \frac{2B - 3C}{2gg}\Big)$$
$$2\beta G' = \tfrac{1}{2}\delta C' + \frac{3A' + 9B'}{8mg} + \frac{1}{4m}\Big(1 + \frac{3A - 9B - 2C}{2gg}\Big)$$
$$4\alpha H' = \frac{3A' + 9B'}{8mg} + \frac{8A - 9B}{8mgg} = 0$$
$$2(2\alpha-\beta)J' = -\frac{9A'}{8mg} - \frac{9A}{8mgg} = 0 \;\Big|\; 2(2\alpha+\beta)K' = -\frac{3B'}{8mg} + \frac{3B}{8mgg} = 0$$

ob $A' = -\frac{A}{g}$; $B' = \frac{B}{g}$ et $C' = \frac{C}{g}$ proxime.

XXXVI.

XXXVI.

Cum autem fit proxime: $\gamma = 2mg$; $\delta = 2mg + \frac{1}{2mg}$; his valoribus quoque fubftitutis fiet :

$$(2\alpha-\beta)\,A' = \frac{9}{4mg}\,;\ (2\alpha+\beta)\,B' = -\frac{3}{4mg}\,;\ \varepsilon C' = \frac{1}{2mg}\,;\ \text{fiue:}$$

$$A' = -\frac{A}{g}\ ;\quad B' = \frac{B}{g}\ ;\quad C' = \frac{C}{g}\,;$$

$$2\alpha D' = \frac{3}{4m} - mA + 3mB\,;$$

$$2(\alpha-\beta)E' = \frac{3}{2m} - mA$$

$$2(\alpha+\beta)F' = -\frac{3}{4m} + 3mB$$

$$2\beta G' = \frac{1}{4m} + mC$$

et reliqui coefficientes H', J', K' pro euanefcentibus funt habendi. Valores autem litterarum A, B, C, etc. §. 25. funt exhibiti.

XXXVII.

Quaeramus nunc quoque longitudinem lunae Φ, huncque in finem fingamus:

$$\Phi = \text{Conft.}$$

$$+\mathfrak{A}'\omega+\mathfrak{B}'\sin v+\mathfrak{C}'\sin 2\eta+\mathfrak{D}'\sin(2\eta-2v)+\mathfrak{E}'\sin(2\eta+2v)+\mathfrak{F}'\sin 2v$$
$$+\mathfrak{G}'\sin(2\eta-v)+\mathfrak{H}'\sin(2\eta+v)+\mathfrak{J}'\sin(2\eta-3v)+\mathfrak{K}'\sin(2\eta+3v)+\mathfrak{L}'\sin 3v$$
$$+\mathfrak{M}'\sin(4\eta-v)+\mathfrak{N}'\sin(4\eta+v)+\mathfrak{P}'\sin(4\eta-3v)+\mathfrak{Q}'\sin(4\eta+3v)$$

ac

ac differentiatione inftituta obtinebimus :

$$\frac{d\Phi}{d\omega} = +\mathfrak{A}' - \tfrac{1}{2}\delta\mathfrak{B}'$$
$$+\ \cos v\left(\beta\mathfrak{B}' - \tfrac{1}{2}\kappa\mathfrak{B}' + \frac{9\mathfrak{D}'}{4mg} + \frac{3\mathfrak{E}'}{4mg} - \delta\mathfrak{F}'\right.$$
$$+\ \cos 2\eta\left(-\frac{3\mathfrak{B}'}{4mg} + 2\alpha\mathfrak{C}'\right.$$
$$+\cos(2\eta-2v)\left(-\frac{9\mathfrak{B}'}{8mg} + 2(\alpha-\beta)\mathfrak{D}'\right.$$
$$+\cos(2\eta+2v)\left(+\frac{3\mathfrak{B}'}{8mg} + 2(\alpha+\beta)\mathfrak{E}'\right.$$
$$+\ \cos 2v\left(-\tfrac{1}{2}\delta\mathfrak{B}' + 2\beta\mathfrak{F}'\right.$$
$$+\cos(2\eta-v)\left(-\tfrac{1}{2}\zeta\mathfrak{B}'-\tfrac{1}{2}\eta\mathfrak{B}'-\gamma\mathfrak{C}'-(\gamma-\delta)\mathfrak{D}'+\frac{3\mathfrak{F}'}{4mg}+(2\alpha-\beta)\mathfrak{G}'\right.$$
$$+\cos(2\eta+v)\left(-\tfrac{1}{2}\zeta\mathfrak{B}'-\tfrac{1}{2}\theta\mathfrak{B}'-\gamma\mathfrak{C}'-(\gamma+\delta)\mathfrak{E}'-\frac{9\mathfrak{F}'}{4mg}+(2\alpha+\beta)\mathfrak{H}'\right.$$
$$+\cos(2\eta-3v)\left(-\tfrac{1}{2}\eta\mathfrak{B}'-(\gamma-\delta)\mathfrak{D}'-\frac{9\mathfrak{F}'}{4mg}+(2\alpha-3\beta)\mathfrak{J}'\right.$$
$$+\cos(2\eta+3v)\left(-\tfrac{1}{2}\theta\mathfrak{B}'-(\gamma+\delta)\mathfrak{E}'+\frac{3\mathfrak{F}'}{4mg}+(2\alpha+3\beta)\mathfrak{K}'\right.$$
$$+\ \cos 3v\left(-\tfrac{1}{2}\kappa\mathfrak{B}'-\frac{3\mathfrak{D}'}{4mg}-\frac{9\mathfrak{E}'}{4mg}-\delta\mathfrak{F}'+3\beta\mathfrak{L}'\right.$$
$$+\cos(4\eta-v)\left(-\frac{3A+9B}{16mgg}\mathfrak{B}'+\frac{9A}{16mgg}\mathfrak{B}'-\frac{3\mathfrak{D}'}{4mg}+(4\alpha-\beta)\mathfrak{M}'\right.$$
$$+\cos(4\eta+v)\left(-\frac{3A+9B}{16mgg}\mathfrak{B}'-\frac{3B}{16mgg}\mathfrak{B}'-\frac{9\mathfrak{E}'}{4mg}+(4\alpha+\beta)\mathfrak{N}'\right.$$
$$+\cos(4\eta-3v)\left(+\frac{9A}{16mgg}\mathfrak{B}'+\frac{9\mathfrak{D}'}{4mg}+(4\alpha-3\beta)\mathfrak{P}'\right.$$
$$+\cos(4\eta+3v)\left(-\frac{3B}{16mgg}\mathfrak{B}'+\frac{3\mathfrak{E}'}{4mg}+(4\alpha+3\beta)\mathfrak{Q}'\right.$$

XXXVIII.

XXXVIII.

Comparata iam hac forma cum valore ipſius $\frac{d\varphi}{d\omega}$ in §. 29. exhibito, obtinebitur

$$\mathfrak{A}' = \tfrac{1}{2}\delta\mathfrak{B}' + m(1 + \tfrac{1}{2}gg - C)$$

$$\beta\mathfrak{B}' = \tfrac{1}{2}\kappa\mathfrak{B}' - \frac{9\mathfrak{D}' - 3\mathfrak{E}'}{4mg} + \delta\mathfrak{F}' - 2mg + \tfrac{3}{2}mgC - mG$$

$$2\alpha\mathfrak{C}' = \frac{3\mathfrak{B}'}{4mg} - m(\tfrac{3}{2}\mathfrak{A} + A + B)$$

$$2(\alpha-\beta)\mathfrak{D}' = \frac{9\mathfrak{B}'}{8mg} - mA$$

$$2(\alpha+\beta)\mathfrak{E}' = -\frac{3\mathfrak{B}'}{8mg} - mB$$

$$2\beta\mathfrak{F}' = \tfrac{1}{2}\delta\mathfrak{B}' - m(C - \tfrac{1}{2}gg)$$

$$(2\alpha-\beta)\mathfrak{G}' = \tfrac{1}{2}(\zeta+\eta)\mathfrak{B}' + \gamma\mathfrak{C}' + (\gamma-\delta)\mathfrak{D}' - \frac{3\mathfrak{F}'}{4mg} + m(\tfrac{3}{2}g\mathfrak{A} - \tfrac{3}{2}\mathfrak{B} + gA + \tfrac{1}{2}gB - D - E)$$

$$(2\alpha+\beta)\mathfrak{H}' = \tfrac{1}{2}(\zeta+\theta)\mathfrak{B}' + \gamma\mathfrak{C}' + (\gamma+\delta)\mathfrak{E}' + \frac{9\mathfrak{F}'}{4mg} + m(\tfrac{3}{2}g\mathfrak{A} - \tfrac{3}{2}\mathfrak{C} + gB + \tfrac{1}{2}gA - D - F)$$

$$(2\alpha-3\beta)\mathfrak{I}' = \tfrac{1}{2}\eta\mathfrak{B}' + (\gamma-\delta)\mathfrak{D}' + \frac{9\mathfrak{F}'}{4mg} + m(\tfrac{1}{2}gA - E)$$

$$(2\alpha+3\beta)\mathfrak{K}' = \tfrac{1}{2}\theta\mathfrak{B}' + (\gamma+\delta)\mathfrak{E}' - \frac{3\mathfrak{F}'}{4mg} + m(\tfrac{1}{2}gB - F)$$

$$3\beta\mathfrak{L}' = \tfrac{1}{2}\kappa\mathfrak{B}' + \frac{3\mathfrak{D}' + 9\mathfrak{E}'}{4mg} + \delta\mathfrak{F}' + m(\tfrac{1}{2}gC - G)$$

$$(4\alpha-\beta)\mathfrak{M}' = -\frac{6A - 9B}{16mgg}\mathfrak{B}' + \frac{3\mathfrak{D}'}{4mg} - m(H + J)$$

$$(4\alpha+\beta)\mathfrak{N}' = +\frac{3A - 6B}{16mgg}\mathfrak{B}' + \frac{9\mathfrak{E}'}{4mg} - m(H + K)$$

$$(4\alpha-3\beta)\mathfrak{P}' = -\frac{9A}{16mgg}\mathfrak{B}' - \frac{9\mathfrak{D}'}{4mg} - mJ$$

$$(4\alpha+3\beta)\mathfrak{Q}' = +\frac{3B}{16mgg}\mathfrak{B}' - \frac{3\mathfrak{E}'}{4mg} - mK$$

XXXIX.

XXXIX.

Inuentis iam valoribus litterarum $p = b(1+\xi)$ et q vna cum anomalia vera v, diſtantia curtata lunae a terra $x = \frac{p}{1 - q\cos v}$ cognoſcetur: ac ſi deinceps latitudinis lunae ψ ratio habebitur, erit diſtantia vera $= \frac{p}{(1 - q\cos v)\cos\psi}$. In Aſtronomia autem non tam diſtantia lunae, quam eius diameter apparens et parallaxis requiri ſolet; quarum vtraque cum ſit diſtantiae lunae a terra reciproce proportionalis, erit tam diameter apparens quam parallaxis vt $\frac{(1 - q\cos v)\cos\psi}{p}$; vnde ſi vtriusque valor medius ex obſeruationibus fuerit definitus, ad quoduis tempus valor verus aſſignari poterit. Sit igitur ſiue diametri apparentis ſiue parallaxis horizontalis valor medius $= \sigma$, eritque is pro tempore dato $= \frac{b\sigma}{p}(1 - q\cos v)\cos\psi$. Eſt autem proxime $\cos\psi = 1 - \tfrac{1}{4}\tan g\rho^2 - \tfrac{1}{4}\mathrm{tg}\,\rho^2\,\mathrm{cs}\,2(\varphi - \pi) = \tfrac{2}{3} + \frac{\lambda + \Pi}{3}$, et $\frac{b}{p} = 1 - \xi + \xi\xi$: vnde fit diameter ſeu parallaxis $= \tfrac{1}{3}\sigma(2 + \lambda + \Pi)(1 - \xi + \xi\xi)(1 - q\cos v)$, quae euoluitur in hanc expreſſionem: ob $\frac{2+\lambda}{3} = 1$ proxime:

$$\tfrac{1}{3}(2+\lambda)\sigma\,[1 - q\cos v - \xi + q\xi\cos v + \xi\xi + \tfrac{1}{3}\Pi - \tfrac{1}{3}q\Pi\cos v]$$

XL.

Pro praeſenti ergo caſu, quo parallaxin ſolis, eiusque excentricitatem vna cum inclinatione orbitae lunaris

ris negligimus, erit lunae diameter apparens vel parallaxis horizontalis

$$= \tfrac{1}{3}(2+\lambda)\sigma \text{ mult. per}$$

$$1 - \tfrac{1}{2}C - (g + \tfrac{1}{2}G)\cos v$$
$$-\tfrac{1}{2}(2\mathfrak{A}+A+B)\cos 2\eta - \tfrac{1}{2}A\cos(2\eta-2v) - \tfrac{1}{2}B\cos(2\eta+2v) - \tfrac{1}{2}C\cos 2v$$
$$-\tfrac{1}{2}(2\mathfrak{B}+D+E)\cos(2\eta-v) - \tfrac{1}{2}(2\mathfrak{C}+D+F)\cos(2\eta+v)$$
$$-\tfrac{1}{2}E\cos(2\eta-3v) - \tfrac{1}{2}F\cos(2\eta+3v) - \tfrac{1}{2}G\cos 3v$$
$$-\tfrac{1}{2}(H+J)\cos(4\eta-v) - \tfrac{1}{2}(H+K)\cos(4\eta+v) - \tfrac{1}{2}J\cos(4\eta-3v) - \tfrac{1}{2}K\cos(4\eta+3v)$$

vbi quidem factor constans $\tfrac{1}{3}(2+\lambda)\sigma$ omitti potest, siquidem tantum proportio vel diametri apparentis vel parallaxis horizontalis desideretur.

XLI.

Si nunc hos valores in numeris euoluere velimus, ex obseruationibus primum colligimus has determinationes:

$\mathfrak{A}' = 13,3682$; $\Delta = 0,1123$ et proxime $g = 0,05445$

ac postremo quidem valore ipsius g tantum in terminis minimis vtar, in maioribus ipsam litteram g relicturus, vt deinceps ex collatione calculi cum obseruationibus accuratius fortasse determinari possit. Habemus ergo

$$13,3682 = \tfrac{1}{2}\delta\mathfrak{B}' + m(1 + \tfrac{1}{2}gg - C)$$

vnde ob $\mathfrak{B}'$, gg et C numeros admodum paruos, statim prope colligitur $m = 13,3682$. Tum vero est prope $\mathfrak{B}' = -\frac{2mg}{\epsilon}$; $\epsilon = m$ et $\delta = 2mg + \frac{1}{2mg}$ seu $\epsilon = 13,3682$; $\delta = 2,$

$\delta = 2,1419$; hinc $\tfrac{1}{2}\delta\mathfrak{B}' = -0,1165$, ergo accuratius $m(1 + \tfrac{1}{2}gg - C) = 13,4847 = \alpha + 1$ et $\alpha = 12,4847$

Porro ob $C = \frac{1}{2mm}$ erit satis exacte . $m = 13,5039$

vnde ex valoribus A et B proxime collectis fit $\epsilon = 13,0644$
$\gamma = 26,9524g.$

At valor ipsius δ duabus constat partibus, altera per g multiplicata altera diuisa, quibus separatim expressis erit

$$\delta = 27,0355\,g + 0,0370.\frac{1}{g} = 2,1521 \text{ proxime.}$$

XLII.

Hinc iam computo instituto sequentes supra assumtorum coefficientium eruuntur valores numerici:

$\mathfrak{A} = 0,008931$; $\mathfrak{B} = 0,03895g = 0,00209$; $\mathfrak{C} = 0,01219g = 0,00064$

ideoque

$$\xi = 0,008931\cos 2\eta + 0,03895\,g\cos(2\eta-v) + 0,01219\,g\cos(2\eta+v)$$

Deinde reperitur:

$A = -0,013995$; $B = -0,001460$; $C = +0,002834$

$D = -0,001213.g + 0,00002198.\frac{1}{g} = -0,000280$ proxime

$E = +0,25834.g - 0,00001889.\frac{1}{g} = +0,014012$ proxime

$F = -0,00225.g - 0,00000207.\frac{1}{g} = -0,000161$ proxime

$G = +0,00218.g + 0,00001221.\frac{1}{g} = +0,000340$ proxime

$H = -0,00001022.\frac{1}{g} = -0,000184$

$J = +0,00004897.\frac{1}{g} = +0,000882$

$K = +0,00000053.\frac{1}{g} = +0,000009$

XLIII.

XLIII.

Hinc porro pro motu apogei eiusque inaequalitatibus colligitur: $\Delta = 0,1123$; qui quidem valor ex obseruationibus est desumtus

$$A' = -0,013995 \cdot \frac{1}{g} = -0,25703 \text{ proxime}$$

$$B' = -0,001460 \cdot \frac{1}{g} = -0,02682 \text{ proxime}$$

$$C' = +0,002834 \cdot \frac{1}{g} = +0,05205 \text{ proxime}$$

$D' = +0,007432$	$F' = -0,002249$
$E' = -0,259170$	$G' = -0,002176$

in minutis secundis

$A' = -53018''$	$= -14°, 43', 38''$
$B' = -5532$	$= -1°, 32, 12$
$C' = +10736$	$= +2, 58, 56$
$D' = +1533$	$= +0, 25, 33$
$E' = -53459$	$= -14, 50, 59$
$F' = -464$	$= -0, 7, 44$
$G' = +449$	$= +0, 7, 29$

Ergo longitudo apogei in minutis secundis

$$\phi - v = \text{Const.}$$

$$+0,1123w - 53018'' \sin(2\eta - v) + 1533'' \sin 2\eta$$
$$- 5532 \sin(2+v) - 53459 \sin(2\eta - 2v)$$
$$+ 10736 \sin v - 464 \sin(2\eta + 2v)$$
$$+ 449 \sin 2v$$

XLIV.

XLIV.

Iam pro longitudine ipsa inuenienda habentur primo ex §. 27. valores: $\gamma = 1,46756$

$\delta = +2,15210$; $\epsilon = -0,027791$; $\zeta = +0,07138$

$\eta = -0,06974$; $\theta = -0,039809$; $\varkappa = -0,070920$

Deinde cum sit proxime

$$6\mathfrak{B}' = -2mg \text{ seu } \mathfrak{B}' = -0,11256$$

erit quoque proxime

$$\mathfrak{C}' = -0,003485 \;;\; \mathfrak{D}' = -0,014456$$

$$\mathfrak{E}' = +0,001509 \;;\; \mathfrak{F}' = -0,005334$$

Hinc ergo accuratius elicietur $\mathfrak{B}' = -0,11019$, ideoque hic et reliqui conficientes tam absolute quam in numeris secundis erunt:

absolute		in minutis secundis	
$\mathfrak{B}' = -0,11019$	. .	$\mathfrak{B}' = -22728''$	$= -6°, 18', 4''$
$\mathfrak{C}' = -0,00339$	. .	$\mathfrak{C}' = -700$	$= -0, 11, 40$
$\mathfrak{D}' = -0,01742$	. .	$\mathfrak{D}' = -3594$	$= -0, 59, 54$
$\mathfrak{E}' = +0,00149$	. .	$\mathfrak{E}' = +306$	$= +0, 5, 6$
$\mathfrak{F}' = -0,00524$	. .	$\mathfrak{F}' = -1081$	$= -0, 18, 1$
$\mathfrak{G}' = -0,01824$	. .	$\mathfrak{G}' = -3762$	$= -1, 2, 42$
$\mathfrak{H}' = -0,00056$	. .	$\mathfrak{H}' = -115$	$= -0, 1, 55$
$\mathfrak{I}' = +0,01368$	. .	$\mathfrak{I}' = +2823$	$= +0, 47, 3$
$\mathfrak{K}' = +0,00023$	. .	$\mathfrak{K}' = +47$	$= +0, 0, 47$
$\mathfrak{L}' = -0,00062$	. .	$\mathfrak{L}' = -128$	$= -0, 2, 8$
$\mathfrak{M}' = -0,00119$	. .	$\mathfrak{M}' = -246$	$= -0, 4, 6$
$\mathfrak{N}' = +0,00020$	. .	$\mathfrak{N}' = +41$	$= +0, 0, 41$
$\mathfrak{P}' = +0,00184$	. .	$\mathfrak{P}' = +379$	$= +0, 6, 19$
$\mathfrak{Q}' = -0,00001$	. .	$\mathfrak{Q}' = -2$	$= -0, 0, 2$

XLV.

XLV.

Hinc ergo fi ad datum tempus iam cognita fit anomalia lunae vera v cum angulo η, longitudo lunae per aequationes in minutis fecundis expreffas erit

$$\Phi = \text{Conft.}$$

$$\begin{array}{llllll}
+13,3682\omega & -22728'' & \sin v & - & 700'' & \sin 2\eta \\
 & -1081 & \sin 2v & - & 3594 & \sin(2\eta-2v) \\
 & -128 & \sin 3v & + & 306 & \sin(2\eta+2v) \\
 & -3762'' & \sin(2\eta-v) & - & 246'' & \sin(4\eta-v) \\
 & -115 & \sin(2\eta+v) & + & 41 & \sin(4\eta+v) \\
 & +2823 & \sin(2\eta-3v) & + & 379 & \sin(4\eta-3v) \\
 & +47 & \sin(2\eta+3v) & - & 2 & \sin(4\eta+3v)
\end{array}$$

vbi Conft. $+13,3682\omega$ denotat longitudinem mediam; in reliquis autem terminis continentur inaequalitates periodicae pro hac hypothefi.

XLVI.

Inde iam viciffim anomalia vera lunae v colligitur, vt fit

$$v =$$

$$\begin{array}{llllll}
13,2559\omega & -33464'' & \sin v & - & 2233'' & \sin 2\eta \\
 & -1530 & \sin 2v & + & 49864 & \sin(2\eta-2v) \\
 & -128 & \sin 3v & + & 770 & \sin(2\eta+2v) \\
 & +49256 & \sin(2\eta-v) & - & 246 & \sin(4\eta-v) \\
 & +5417 & \sin(2\eta+v) & + & 41 & \sin(4\eta+v) \\
 & +2823 & \sin(2\eta-3v) & + & 379 & \sin(4\eta-3v) \\
 & +47 & \sin(2\eta+3v) & - & 2 & \sin(4\eta+3v)
\end{array}$$

vbi primus terminus $13,2559\omega$ defignat anomaliam mediam lunae, quae fit $=\zeta$: tum ex ea primum quaeratur ano-

fit ano-
nae per
2η
2η−2v)
2η+2v)
4η−v)
4η+v)
4η−3v)
4η+3v)
ediam;
tes pe-
ligitur,
2η−2v)
η+2v)
4η−v)
η+v)
η−3v)
η+3v)
m me-
ıeratur
ano-

anomalia Kepleriana, quae fcilicet a fola excentricitate pendet, fitque ea $= s$, vt fit

$$s = \zeta - 33464'' \sin s - 1530'' \sin 2s - 128'' \sin 3s$$

vnde quidem facile tabulae conftruentur. Tum ftatuatur $v = s + z$, et quia angulus z eft modicus, inde is fatis prope poterit definiri. Interim tamen expedire videtur aliquot operationibus iterandis iftam anomaliam veram v determinari; dum fcilicet primum valor non nimis a veritate abhorrens pro v aeftimando affumitur, ex eoque deinceps exactior colligitur; qui fi nimis ab affumto difcrepare reperiatur, ex hoc denuo exactior quaeratur, donec nulla amplius correctione fuerit opus.

XLVII.

Formula denique, cui tam diameter lunae apparens geocentrica quam parallaxis horizontalis eft proportionalis, ex §. 40. reperitur

$$\begin{array}{llll}
1 & -0,05470 \cos v & - & 0,00120 \cos 2\eta \\
 & -0,00142 \cos 2v & + & 0,00700 \cos(2\eta-2v) \\
 & -0,00017 \cos 3v & + & 0,00073 \cos(2\eta+2v) \\
 & -0,00898 \cos(2\eta-v) & - & 0,00035 \cos(4\eta-v) \\
 & -0,00042 \cos(2\eta+v) & + & 0,00009 \cos(4\eta+v) \\
 & -0,00701 \cos(2\eta-3v) & - & 0,00044 \cos(4\eta-3v) \\
 & +0,00008 \cos(2\eta+3v) & - & 0,00001 \cos(4\eta+3v)
\end{array}$$

quorum quidem terminorum plures, qui pro parallaxi infra aliquot minuta fecunda fubfiftunt, tuto omitti poterunt. His igitur tribus formulis pro anomalia vera v, longitudine Φ et parallaxi feu diametro apparente inuentis motus lunae contineretur, fi quidem tam folis parallaxis

laxis quam eius excentricitas et inclinatio orbitae lunaris ad eclipticam negligatur. Hae autem ſunt inaequalitates praecipuae, quae etiam ad reliquas eruendas adhiberi debent; vnde nunc ad inaequalitates ab excentricitate ſolis oriundas progrediamur.

INUESTIGATIO INAEQUALITATUM LUNAE SECUNDI GENERIS SEU AB EXCENTRICITATE SOLIS PENDENTIUM.

XLVIII.

Formulae noſtrae differentiales, quatenus ab excentricitate orbitae ſolaris pendent, omiſſis terminis, quos iam conſtat eſſe minimos, erunt

$$\frac{d\xi}{d\omega} = \text{Praec.} + \frac{9e}{2m}\,\text{ſin}\,(2\eta - u) + \frac{9e}{2m}\,\text{ſin}\,(2\eta + u)$$

$$\frac{dq}{d\omega} = \text{Praec.} + \frac{3e}{4m}\,\text{ſin}\,(v-u) + \frac{3e}{4m}\,\text{ſin}\,(v+u)$$
$$- \frac{27e}{8m}\,\text{ſin}\,(2\eta - v - u) - \frac{27e}{8m}\,\text{ſin}\,(2\eta - v + u)$$
$$- \frac{9e}{8m}\,\text{ſin}\,(2\eta + v - u) - \frac{9e}{8m}\,\text{ſin}\,(2\eta + v + u)$$

$$\frac{q(d\Phi - dv)}{d\omega} = \text{Pr.} - \frac{3e}{4m}\,\text{coſ}\,(v-u) - \frac{3e}{4m}\,\text{coſ}\,(v+u)$$
$$- \frac{27e}{8m}\,\text{coſ}\,(2\eta - v - u) - \frac{27e}{8m}\,\text{coſ}\,(2\eta - v + u)$$
$$+ \frac{9e}{8m}\,\text{coſ}\,(2\eta + v - u) + \frac{9e}{8m}\,\text{coſ}\,(2\eta + v + u)$$

Quan-

Quanquam enim nunc tam ξ quam q etiam ab excentricitate e pendeant, tamen in his formulis, in quas hae quantitates ingrediuntur, haec mutatio earum ſine errore pro nihilo haberi poteſt; quoniam hi termini per ſe ſunt minimi, et quia iam terminos ab e et q ſimul pendentes omiſimus. Tum vero erit

$$\frac{d\Phi}{d\omega} = m(1 + \tfrac{1}{4}qq) - 2mq\,\text{coſ}\,v + \tfrac{1}{2}mqq\,\text{coſ}\,2v - \tfrac{3}{2}m\xi + 3mq\xi\,\text{coſ}\,v$$

$$\text{et } \frac{du}{d\omega} = \frac{d\theta}{d\omega} = 1 + 2ee - 2e\,\text{coſ}\,u$$

XLIX.

Ad formulas has integrandas ſeu tantum ad eas integralium partes inueniendas, quae ab excentricitate ſolis e pendent, opus eſt vt formularum $\frac{d\eta}{d\omega}$, $\frac{dv}{d\omega}$ et $\frac{du}{d\omega}$ primum habeamus partes principales, tum vero etiam eas quae a ſimplici ſolis excentricitate e pendent: habebimus ergo primo

$$\frac{d\eta}{d\omega} = m(1 + \tfrac{1}{4}gg) - 1 - 2ee - 2mg\,\text{coſ}\,v + 2e\,\text{coſ}\,u$$

$$\frac{dv}{d\omega} = m(1 + \tfrac{1}{4}gg) - 2mg\,\text{coſ}\,v + \frac{3e}{4mg}\,\text{coſ}(v-u) + \frac{3e}{4mg}\,\text{coſ}(v+u)$$
$$- \frac{1}{4m}\left(1 - \frac{9A + 3B - 2C}{2gg}\right) + \frac{27e}{8mg}\,\text{cſ}(2\eta - v - u) + \frac{27e}{8mg}\,\text{cſ}(2\eta - v + u)$$
$$- \frac{9e}{8mg}\,\text{cſ}(2\eta + v - u) - \frac{9e}{8mg}\,\text{cſ}(2\eta + v + u)$$

ſeu introducendis, vt ſupra §. 27. breuitatis gratia, litteris

$$\alpha = m(1 + \tfrac{1}{4}gg - C) - 2ee \quad ; \quad \gamma = 2mg$$

$$\beta = m(1 + \tfrac{1}{4}gg - C) - \frac{1}{4m}\left(1 - \frac{9A + 3B - 2C}{2gg}\right); \quad \delta = 2mg + \frac{1}{2mg} + \frac{3g}{8m};$$

erit:

erit: $\frac{d\eta}{d\omega} \alpha - \gamma \cos v + 2e \cos u$

$$\frac{dv}{d\omega} = \beta - \delta \cos v - \frac{9}{4mg} \cos(2\eta - v) + \frac{3}{4mg} \cos(2\eta + v)$$
$$+ \frac{3e}{4mg} \cos(v-u) + \frac{3e}{4mg} \cos(v+u) + \frac{27e}{8mg} \cos(2\eta - v - u)$$
$$+ \frac{27e}{8mg} \cos(2\eta - v + u) - \frac{9e}{8mg} \cos(2\eta + v - u) - \frac{9e}{8mg} \cos(2\eta + v + u)$$

et $\frac{du}{d\omega} = 1 - 2e \cos u$.

L.

Fingamus nunc primo:

$$\xi = \mathfrak{A} \cos 2\eta + \mathfrak{B} \cos(2\eta - v) + \mathfrak{C} \cos(2\eta + v) + \mathfrak{P} \cos(2\eta - u) + \mathfrak{Q} \cos(2\eta + u)$$

ac differentiando eos tantum sumamus terminos, qui formulae differentiali respondent, quandoquidem reliquos iam inuenimus: eritque

$$\frac{d\xi}{d\omega} = -2e\mathfrak{A} \sin(2\eta - u) - 2e\mathfrak{A} \sin(2\eta + u)$$
$$- (2\alpha - 1)\mathfrak{P} \qquad - (2\alpha + 1)\mathfrak{Q}$$

vnde colligitur:

$$(2\alpha - 1)\mathfrak{P} = -\frac{9e}{2m} - 2e\mathfrak{A}; \quad (2\alpha + 1)\mathfrak{Q} = -\frac{9e}{2m} - 2e\mathfrak{A}$$

Cum igitur sit $e = 0,0168$, erit in numeris:

$$\mathfrak{P} = -0,000247 \quad \text{et} \quad \mathfrak{Q} = -0,000227.$$

LI.

Fingatur porro: $q = g$

$$+ A\cos(2\eta - v) + B\cos(2\eta + v)\, C\cos v + M\cos(v-u) + N\cos(v+u)$$
$$+ P\cos(2\eta - v - u) + Q\cos(2\eta - v + u) + R\cos(2\eta + v - u) + S\cos(2\eta + v + u)$$

ac

ac differentiando obtinebitur: $\frac{dq}{d\omega} =$

$$-2eA\sin(2\eta - v - u) - 2eA\sin(2\eta - v + u) - 2eB\sin(2\eta + v - u) - 2eB\sin(2\eta + v + u)$$
$$-(2\alpha - \beta - 1)P \quad -(2\alpha - \beta + 1)Q \quad -(2\alpha + \beta - 1)R \quad -(2\alpha + \beta + 1)S$$
$$-(\beta - 1) M \sin(v - u) - (\beta + 1) H \sin(v + u)$$

Comparatione ergo instituta reperietur:

$$(\beta - 1) M = -\frac{3e}{4m}; \quad (\beta + 1) N = -\frac{3e}{4m}$$
$$(2\alpha - \beta - 1) P = \frac{27e}{8m} - 2eA; \quad (2\alpha - \beta + 1) Q = \frac{27e}{8m} - 2eA$$
$$(2\alpha + \beta - 1) R = \frac{9e}{8m} - 2eB; \quad (2\alpha + \beta + 1) S = \frac{9e}{8m} - 2eB$$

et in numeris

$M = -0,00008$; $P = +0,00042$; $R = +0,00004$

$N = -0,00006$; $Q = +0,00036$; $S = +0,00004$

LII.

Hic autem in differentiatione neglesimus partes ipsius $\frac{dv}{d\omega}$ ab excentricitate e pendentes, quarum tamen eodem iure ratio haberi debuisset, atque partis in differentiali $\frac{d\eta}{d\omega}$; inde autem multo plures termini accedent ad valorum ipsius q, ponatur ergo ob hos terminos:

$$q = g$$
$$+ A\cos(2\eta - v) + B\cos(2\eta + v) + C\cos v + M\cos(v-u)$$
$$+ N\cos(v+u) + D\cos(2\eta - u) + E\cos(2\eta + u)$$
$$+ P\cos(2\eta - v - u) + Q\cos(2\eta - v + u) + R\cos(2\eta + v - u)$$
$$+ S\cos(2\eta + v + u) + K\cos(2v - u) + L\cos(2v + u)$$

+

$$+F\cos(2\eta-2v-u)+G\cos(2\eta-2v+u)+H\cos(2\eta+2v-u)$$
$$+J\cos(2\eta+2v+u)+T\cos(4\eta-u)+V\cos(4\eta+u)$$
$$+W\cos(4\eta-2v-u)+X\cos(4\eta-2v+u)+Y\cos(4\eta+2v-u)$$
$$+Z\cos(4\eta+2v+u)$$

et sumto differentiali pleno reperiotur:

$$\frac{dq}{d\omega}=+\sin(2\eta-v-u)\,[-2eA-(2\alpha-\beta-1)P]$$
$$+\sin(2\eta-v+u)\,[-2eA-(2\alpha-\beta+1)Q]-(\beta-1)M\sin(v-u)$$
$$+\sin(2\eta\mp v-u)\,[-2eB-(2\alpha+\beta-1)R]$$
$$+\sin(2\eta+v+u)\,[-2eB-(2\alpha+\beta+1)S]-(\beta+1)N\sin(v+u)$$
$$+\sin(2\eta-u)\left(+\frac{3eA}{8mg}-\frac{3eB}{8mg}-\frac{27eC}{16mg}-\frac{9eC}{16mg}-(2\alpha-1)D\right)$$
$$+\sin(2\eta+u)\left(+\frac{3eA}{8mg}-\frac{8eB}{8mg}-\frac{27eC}{16mg}-\frac{9eC}{16mg}-(2\alpha+1)E\right)$$
$$+\sin(2\eta-2v+u)\left(+\frac{3eA}{8mg}+\frac{27eC}{16mg}-(2\alpha-2\beta+1)G\right)$$
$$+\sin(2\eta-2v-u)\left(+\frac{3eA}{8mg}+\frac{27eC}{16mg}-(2\alpha-2\beta-1)F\right)$$
$$+\sin(2\eta+2v-u)\left(-\frac{3eB}{8mg}+\frac{9eC}{16mg}-(2\alpha+2\beta-1)H\right)$$
$$+\sin(2\eta+2v+u)\left(-\frac{3eB}{8mg}+\frac{9eC}{16mg}-(2\alpha+2\beta+1)J\right)$$
$$+\sin u\left(+\frac{27eA}{16mg}-\frac{27eA}{16mg}+\frac{9eB}{16mg}-\frac{9eB}{16mg}-\frac{3eC}{8mg}+\frac{3eC}{8mg}\right)$$
$$+\sin(4\eta-2v-u)\left(+\frac{27eA}{16mg}-(4\alpha-2\beta-1)W\right)$$
$$+\sin(4\eta-2v+u)\left(+\frac{27eA}{16mg}-(4\alpha-2\beta+1)X\right)$$

+

$$+\sin(4\eta+2v-u)\left(+\frac{9eB}{16mg}-(4\alpha+2\beta-1)Y\right)$$
$$+\sin(4\eta+2v+u)\left(+\frac{9eB}{16mg}-(4\alpha+2\beta+1)Z\right)$$
$$+\sin(2v-u)\left(+\frac{9eA}{16mg}-\frac{27eB}{16mg}-\frac{3eC}{8mg}-(2\beta-1)K\right)$$
$$+\sin(2v+u)\left(+\frac{9eA}{16mg}-\frac{27eB}{16mg}-\frac{3eC}{8mg}-(2\beta+1)L\right)$$
$$+\sin(4\eta-u)\left(-\frac{9eA}{16mg}-\frac{27eB}{16mg}-(4\alpha-1)T\right)$$
$$+\sin(4\eta+u)\left(-\frac{9eA}{16mg}-\frac{27eB}{16mg}-(4\alpha+1)V\right)$$

vnde reperitur:

$D=-0{,}000010$;	$H=+0{,}000001$
$E=-0{,}000010$;	$J=+0{,}000001$
$F=+0{,}000005$;	$K=-0{,}000006$
$G=+0{,}000065$;	$L=-0{,}000006$
$T=+0{,}000004$;	$X=-0{,}000022$
$V=+0{,}000004$;	$Y=-0{,}000000$
$W=-0{,}000023$;	$Z=-0{,}000000$

L'II.

Ponamus nunc etiam pro motu apogei

$$\Phi-v=\text{Const.}+\Delta\omega$$
$$+A'\sin(2\eta-v)+B'\sin(2\eta+v)+C'\sin v+M'\sin(v-u)+N'\sin(v+u)$$
$$+P'\sin(2\eta-v-u)+Q'\sin(2\eta-v+u)+R'\sin(2\eta+v-u)+S'\sin(2\eta+v+u)$$
$$+D'\sin(2\eta-u)+E'\sin(2\eta+u)+K'\sin(2v-u)+L'\sin(2v+u)+O'\sin u$$
$$+F'\sin(2\eta-2v-u)+G'\sin(2\eta-2v+u)+H'\sin(2\eta+2v-u)+J'\sin(2\eta+2v+u)$$
$$+W'\sin(4\eta-2v-u)+X'\sin(4\eta-2v+u)+Y'\sin(4\eta+2v-u)+Z'\sin(4\eta+2v+u)$$
$$+T'\sin(4\eta-u)+V'\sin(4\eta+u)$$

et fumto differentiali pleno reperietur:

$$\frac{d\Phi-dv}{d\omega} = \Delta + \mathrm{cof}(2\eta-v-u)\ [2eA'+(2\alpha-\beta-1)P']$$
$$+\mathrm{cof}(2\eta-v+u)\ [2eA'+(2\alpha-\beta+1)Q'] + (\beta-1)M'\mathrm{cof}(v-u)$$
$$+\mathrm{cof}(2\eta+v-u)\ [2eB'+(2\alpha+\beta-1)R']$$
$$+\mathrm{cof}(2\eta+v+u)\ [2eB'+(2\alpha+\beta+1)S'] + (\beta+1)N'\mathrm{cof}(v+u)$$
$$+\mathrm{cf}(2\eta-u)\left(-\frac{3e}{8mg}A'+\frac{3e}{8mg}B'+\frac{27e}{16mg}C'-\frac{9e}{16mg}C'+(2\alpha-1)D'\right)$$
$$+\mathrm{cf}(2\eta+u)\left(-\frac{3e}{8mg}A'+\frac{3e}{8mg}B'+\frac{27e}{16mg}C'-\frac{9e}{16mg}C'+(2\alpha+1)E'\right)$$
$$+\mathrm{cof}(2\eta-2v-u)\left(-\frac{3e}{8mg}A'+\frac{27e}{16mg}C'+(2\alpha-2\beta-1)F'\right)$$
$$+\mathrm{cof}(2\eta-2v+u)\left(-\frac{3e}{8mg}A'+\frac{27e}{16mg}C'+(2\alpha-2\beta+1)G'\right)$$
$$+\mathrm{cof}(2\eta+2v-u)\left(+\frac{3e}{8mg}B'+\frac{9e}{16mg}C'+(2\alpha+2\beta-1)H'\right)$$
$$+\mathrm{cof}(2\eta+2v+u)\left(+\frac{3e}{8mg}B'+\frac{9e}{16mg}C'+(2\alpha+2\beta+1)J'\right)$$
$$+\ \mathrm{cof}\,u\left(-\frac{27e}{16mg}A'-\frac{27e}{16mg}A'-\frac{9e}{16mg}B'-\frac{9e}{16mg}B' + \frac{3e}{8mg}C'+\frac{3e}{8mg}C'+O'\right)$$
$$+\mathrm{cof}(4\eta-2v-u)\left(-\frac{27e}{16mg}A'+(4\alpha-2\beta-1)W'\right)$$
$$+\mathrm{cof}(4\eta-2v+u)\left(-\frac{27e}{16mg}A'+(4\alpha-2\beta+1)X'\right)$$
$$+\mathrm{cof}(4\eta+2v-u)\left(-\frac{9e}{16mg}B'+(4\alpha+2\beta-1)Y'\right)$$
$$+\mathrm{cof}(4\eta+2v+u)\left(-\frac{9e}{16mg}B'+(4\alpha+2\beta+1)Z'\right)$$

$$+\mathrm{cof}(2v-u)\left(+\frac{9e}{16mg}A'+\frac{27e}{16mg}B'\mp\frac{3e}{8mg}C'+(2\beta-1)K'\right)$$
$$+\mathrm{cof}(2v+u)\left(+\frac{9e}{16mg}A'+\frac{27e}{16mg}B'+\frac{3e}{8mg}C'+(2\beta+1)L'\right)$$
$$+\mathrm{cof}(4\eta-u)\left(+\frac{9e}{16mg}A'+\frac{27e}{16mg}B'+(2\alpha-1)T'\right)$$
$$+\mathrm{cof}(4\eta+u)\left(+\frac{9e}{16mg}A'+\frac{27e}{16mg}B'+(2\alpha+1)V'\right)$$

LIV.

Singuli iam hi termini multiplicentur per q, cuius valor quidem erit $= g$, quoniam hi termini in fuo genere iam funt minimi: fed quoniam valor $\frac{d\Phi-dv}{d\omega}$ adhuc hos terminos praecipuos continet:

$$(2\alpha-\beta)A'\mathrm{cof}(2\eta-v) + (2\alpha+\beta)B'\mathrm{cof}(2\eta+v) + \beta C'\mathrm{cof}\,v$$

fi et hi per q multiplicentur, inde nafcentur quoque termini angulum u inuoluentem, erit autem pro his, fumtis partibus tantum praecipuis:

$$q = \text{Praec.} + P\,\mathrm{cof}(2\eta-v-u) + Q\,\mathrm{cof}(2\eta-v+u)$$

Ergo ad illos terminos per q multiplicatos infuper accedent ifti:

$\mathrm{cof}\,u\ [\tfrac{1}{2}(2\alpha-\beta)PA'+\tfrac{1}{2}(2\alpha-\beta)QA']$ $\quad + \tfrac{1}{2}\beta PC'\mathrm{cof}(2\eta-u)$
$\mathrm{cof}(4\eta-2v-u)\ [\tfrac{1}{2}(2\alpha-\beta)PA'+\tfrac{1}{2}\beta PC']$ $\quad + \tfrac{1}{2}\beta QC'\mathrm{cof}(2\eta+u)$
$\mathrm{cof}(4\eta-2v+u)\ [\tfrac{1}{2}(2\alpha-\beta)QA'+\tfrac{1}{2}\beta QC']$
$\mathrm{cof}(2v+u)\ [\tfrac{1}{2}(2\alpha+\beta)PB'$ $\quad + \mathrm{cof}(4\eta-u)\ [\tfrac{1}{2}(2\alpha+\beta)PB'$
$\mathrm{cof}(2v-u)\ [\tfrac{1}{2}(2\alpha+\beta)QB'$ $\quad + \mathrm{cof}(4\eta+u)\ [\tfrac{1}{2}(2\alpha+\beta)QB'$

LV.

Hinc ergo obtinentur fequentes determinationes:

$$2eg\,A' + (2\alpha-\beta-1)\,g\,P' = -\frac{27e}{8m}$$

$$2eg\,A' + (2\alpha-\beta+1)\,g\,Q' = -\frac{27e}{8m}$$

$$2eg\,B' + (2\alpha+\beta-1)\,g\,R' = +\frac{9e}{8m}$$

$$2eg\,B' + (2\alpha+\beta+1)\,g\,S' = +\frac{9e}{8m}$$

$$(\beta-1)\,g\,M' = -\frac{3e}{4m} \quad ; \quad (\beta+1)\,g\,N' = -\frac{3e}{4m}$$

$$-\frac{3e}{8m}(A'-B') + \frac{9e}{8m}C' + (2\alpha-1)\,g\,D' + \tfrac{1}{2}\beta PC' = 0$$

$$-\frac{3e}{8m}(A'-B') + \frac{9e}{8m}C' + (2\alpha+1)\,g\,E' + \tfrac{1}{2}\beta QC' = 0$$

$$-\frac{3e}{8m}A' + \frac{27e}{16m}C' + (2\alpha-2\beta-1)\,g\,F' = 0$$

$$-\frac{3e}{8m}A' + \frac{27e}{16m}C' + (2\alpha-2\beta+1)\,g\,G' = 0$$

$$+\frac{3e}{8m}B' + \frac{9e}{16m}C' + (2\alpha+2\beta-1)\,g\,H' = 0$$

$$+\frac{3e}{8m}B' + \frac{9e}{16m}C' + (2\alpha+2\beta+1)\,g\,J' = 0$$

$$-\frac{27e}{8m}A' - \frac{9e}{8m}B' + \frac{3e}{8m}C' + g\,O' + \tfrac{1}{2}(2\alpha-\beta)(P+Q)A' = 0$$

$$-\frac{27e}{8m}A' + (4\alpha-2\beta-1)\,g\,W' + \tfrac{1}{2}(2\alpha-\beta)PA' + \tfrac{1}{2}\beta PC' = 0$$

$$-\frac{27e}{8m}A' + (4\alpha-2\beta+1)\,g\,X' + \tfrac{1}{2}(2\alpha\ \beta)QA' + \tfrac{1}{2}\beta QC' = 0$$

$$-\frac{9e}{16m}B' + (4\alpha+2\beta-1)gY' = 0 \quad ; \quad -\frac{9e}{16m}B' + (4\alpha+2\beta+1)gZ' = 0$$

+

$$+\frac{9e}{16m}A' + \frac{27e}{16m}B' + \frac{3e}{8m}C' + (2\beta-1)gK' + \tfrac{1}{2}(2\alpha+\beta)QB' = 0$$

$$+\frac{9e}{16m}A' + \frac{27e}{16m}B' + \frac{3e}{8m}C' + (2\beta+1)gL' + \tfrac{1}{2}(2\alpha+\beta)PB' = 0$$

$$+\frac{9e}{16m}A' + \frac{27e}{16m}B' + (4\alpha-1)\,g\,T' + \tfrac{1}{2}(2\alpha+\beta)PB' = 0$$

$$+\frac{9e}{16m}A' + \frac{27e}{16m}B' + (4\alpha+1)\,g\,V' + \tfrac{1}{2}(2\alpha+\beta)QB' = 0$$

LVI.

Valores ergo horum coefficientium iam ad minuta fecunda reductorum erunt: $O' = +310''$

$P' = -1285''$; $M' = -293''$; $F' = +401''$
$Q' = -1087$; $N' = -251$; $G' = +5412$
$R' = +148$; $D' = -52$; $H' = -2$
$S' = +141$; $E' = -45$; $J' = -2$
$W' = -91''$; $K' = +61''$
$X' = -95$; $L' = +61$
$Y' = -1$; $T' = +35$
$Z' = -1$; $V' = +31$

Vnde ob excentricitatem orbitae folaris erit:

$\xi = +0{,}008931 \cos 2\eta$ $-0{,}000247 \cos(2\eta-u)$
$+0{,}002090 \cos(2\eta-v)$ $-0{,}000227 \cos(2\eta+u)$
$+0{,}000640 \cos(2\eta+v)$

$\psi = g - 0{,}013595 \cos(2\eta-v)$ $-0{,}000280 \cos 2\eta$
$-0{,}001460 \cos(2\eta+v)$ $+0{,}014012 \cos(2\eta-2v)$
$+0{,}002834 \cos v$ $-0{,}000162 \cos(2\eta+2v)$
$+0{,}000340 \cos 2v$
$-0{,}000184 \cos 4\eta$ $+0{,}000420 \cos(2\eta-v-u)$
$+0{,}000882 \cos(4\eta-2v)$ $+0{,}000360 \cos(2\eta-v+u)$
$+0{,}00000? \cos(4\eta+2v)$

$$\Phi - v = \text{Conft.} + 0,1123\,\omega$$

$$-53018''\text{fin}(2\eta-v) + 1533''\text{fin}\,2\eta \quad -1285''\text{fin}(2\eta-v-u)$$
$$-5532\ \text{fin}(2\eta+v) - 53549\ \text{fin}(2\eta-2v) - 1087\ \text{fin}(2\eta-v-u)$$
$$+10736\ \text{fin}\,v \quad -464\ \text{fin}(2\eta+2v) + 148\ \text{fin}(2\eta+v-u)$$
$$+449\ \text{fin}\,2v \quad +141\ \text{fin}(2\eta+v+u)$$
$$-293''\text{fin}(v-u) + 401''\text{fin}(2\eta+2v-u) + 61''\text{fin}(2v-u)$$
$$-251\ \text{fin}(v+u) + 5412\ \text{fin}(2\eta-2v+u) + 61\ \text{fin}(2v+u)$$
$$-52\ \text{fin}(2\eta-u) - 91\ \text{fin}(4\eta-2v-u) + 35\ \text{fin}(4\eta-u)$$
$$-45\ \text{fin}(2\eta+u) - 95\ \text{fin}(4\eta-2v+u) + 31\ \text{fin}(4\eta+u)$$
$$+310\ \text{fin}\,u$$

neglectis fcilicet terminis minimis.

LVII.

Denique pro longitudine lunae vera Φ inuenienda, cum fit $\frac{d\Phi}{d\omega} = \text{Praec.}$

$$+mg.\ 0,00005\,\text{cof}(2\eta-v-u) \quad -m.\ 0,00005\,\text{cof}(2\eta-u)$$
$$+mg.\ 0,00002\,\text{cof}(2\eta-v+u) \quad -m.\ 0,00002\,\text{cof}(2\eta+u)$$
$$+mg.\ 0,00005\,\text{cof}(2\eta+v-u) \quad -m.\ 0,00042\,\text{cof}(2\eta-2v-u)$$
$$+mg.\ 0,00002\,\text{cof}(2\eta+v+u) \quad -m.\ 0,00036\,\text{cof}(2\eta-2v+u)$$
$$+mg.\ 0,00042\,\text{cof}(2\eta-3v-u)$$
$$+mg.\ 0,00036\,\text{cof}(2\eta-3v+u)$$

ponatur $\Phi = \text{Conft.}$

$$+\mathfrak{A}\,\omega + \mathfrak{B}'\text{fin}\,v + \mathfrak{D}'\text{fin}(2\eta-2v) + \mathfrak{G}'\text{fin}(2\eta-v) + \mathfrak{J}'\text{fin}(2\eta-3v)$$

vna cum nouis terminis

$$+\mathfrak{a}'\text{fin}(2\eta-v-u) + \mathfrak{e}'\text{fin}(2\eta-u) + \mathfrak{g}'\text{fin}(2\eta-2v-u) + \mathfrak{l}'\text{fin}\,u$$
$$+\mathfrak{b}'\text{fin}(2\eta-v+u) + \mathfrak{f}'\text{fin}(2\eta+u) + \mathfrak{h}'\text{fin}(2\eta-2v+u) + \mathfrak{m}'\text{fin}(2v-u)$$
$$+\mathfrak{c}'\text{fin}(2\eta+v-u) \quad + \mathfrak{i}'\text{fin}(2\eta-3v-u) + \mathfrak{n}'\text{fin}(2v+u)$$
$$+\mathfrak{d}'\text{fin}(2\eta+v+u) \quad + \mathfrak{k}'\text{fin}(2\eta-3v+u)$$

$+\mathfrak{o}'$

$$+\mathfrak{o}'\text{fin}(2\eta+2v-u) + \mathfrak{q}'\text{fin}(v-u) + \mathfrak{s}'\text{fin}(4v-u)$$
$$+\mathfrak{p}'\text{fin}(2\eta+2v+u) + \mathfrak{r}'\text{fin}(v+u) + \mathfrak{t}'\text{fin}(4v+u)$$

pro reliquorum terminorum, quos forma differentialis requirit, coefficientibus ponamus litteram I.

LVIII.

Differentiatione iam per regulas praecedentes instituta erit: $\frac{d\Phi}{d\omega} = \text{Praec.}$

$$+\text{cof}\,u\left(\frac{3e}{8mg}\mathfrak{B}' + \frac{3e}{8mg}\mathfrak{B}' - \frac{27e}{16mg}\mathfrak{G}' - \frac{27e}{16mg}\mathfrak{G}' + \mathfrak{l}'\right)$$
$$+(2\alpha+\beta-1)\mathfrak{c}'\,\text{cof}(2\eta+v-u)$$
$$+\text{cof}(2v-u)\left(\frac{3e}{8mg}\mathfrak{B}' + \frac{9e}{16mg}\mathfrak{G}' - \frac{18e}{16mg}\mathfrak{J}' + (2\beta-1)\mathfrak{m}'\right)$$
$$+(2\alpha+\beta+1)\mathfrak{d}'\,\text{cof}(2\eta+v+u)$$
$$+\text{cof}(2v+u)\left(\frac{3e}{8mg}\mathfrak{B}' + \frac{9e}{16mg}\mathfrak{G}' - \frac{18e}{16mg}\mathfrak{J}' + (2\beta+1)\mathfrak{n}'\right)$$
$$+\text{cof}(2\eta-u)\left(+\frac{27e}{16mg}\mathfrak{B}' - \frac{9e}{16mg}\mathfrak{B}' - \frac{3e}{8mg}\mathfrak{G}' + (2\alpha-1)\mathfrak{e}'\right)$$
$$+\text{cof}(2\eta+u)\left(+\frac{27e}{16mg}\mathfrak{B}' - \frac{9e}{16mg}\mathfrak{B}' - \frac{3e}{8mg}\mathfrak{G}' + (2\alpha+1)\mathfrak{f}'\right)$$
$$+\text{cof}(2\eta-2v-u)\left(+\frac{27e}{16mg}\mathfrak{B}' + 2e\mathfrak{D}' - \frac{9e}{8mg}\mathfrak{G}' - \frac{9e}{8mg}\mathfrak{J}' + (2\alpha-2\beta-1)\mathfrak{g}'\right)$$
$$+\text{cof}(2\eta-2v+u)\left(+\frac{27e}{16mg}\mathfrak{B}' + 2e\mathfrak{D}' - \frac{3e}{8mg}\mathfrak{G}' - \frac{9e}{8mg}\mathfrak{J}' + (2\alpha-2\beta+1)\mathfrak{h}'\right)$$

+

$$+\cos(4v-u)\left(+\frac{27e}{16mg}\mathfrak{J}'+(4\beta-1)\mathfrak{s}'\right)$$

$$+\cos(4v+u)\left(+\frac{27e}{16mg}\mathfrak{J}'+(4\beta+1)\mathfrak{t}'\right)$$

$$+\cos(2\eta+2v-u)\left(-\frac{9e}{16mg}\mathfrak{B}'+(2\alpha+2\beta-1)\mathfrak{v}'\right)$$

$$+\cos(2\eta+2v+u)\left(-\frac{9e}{16mg}\mathfrak{B}'+(2\alpha+2\beta+1)\mathfrak{y}'\right)$$

$$+\cos(2\eta-v-u)\left(-\frac{3e}{4mg}\mathfrak{D}'+2e\mathfrak{G}'+(2\alpha-\beta-1)\mathfrak{a}'\right)$$

$$+\cos(2\eta-3v+u)\left(-\frac{3e}{8mg}\mathfrak{D}'+2e\mathfrak{J}'+(2\alpha-3\beta+1)\mathfrak{f}'\right)$$

$$+\cos(2\eta-v+u)\left(-\frac{3e}{4mg}\mathfrak{D}'+2e\mathfrak{G}'+(2\alpha-\beta+1)\mathfrak{b}'\right)$$

$$+\cos(2\eta-3v-u)\left(-\frac{3e}{4mg}\mathfrak{D}'+2e\mathfrak{J}'+(2\alpha-3\beta-1)\mathfrak{j}'\right)$$

$$+\cos(v-u)\left(-\frac{27e}{8mg}\mathfrak{D}'+(\beta-1)\mathfrak{q}'\right)$$

$$+\cos(v+u)\left(-\frac{27e}{8mg}\mathfrak{D}'+(\beta+1)\mathfrak{r}'\right)$$

$$+\cos(4\eta-3v-u)\left(-\frac{27e}{8mg}\mathfrak{D}'+(4\alpha-3\beta-1)\mathfrak{l}\right)$$

$$+\cos(4\eta-3v+u)\left(-\frac{27e}{8mg}\mathfrak{D}'+(4\alpha-3\beta+1)\mathfrak{l}\right)$$

$$+\cos(3v-u)\left(+\frac{9e}{8mg}\mathfrak{D}'+(3\beta-1)\mathfrak{l}\right)$$

$$+\cos(3v+u)\left(+\frac{9e}{8mg}\mathfrak{D}'+(3\beta+1)\mathfrak{l}\right)$$

$$+\cos(4\eta-v-u)\left(+\frac{9e}{8mg}\mathfrak{D}'+(4\alpha-\beta-1)\mathfrak{l}\right)$$

$$+\cos(4\eta-v+u)\left(+\frac{9e}{8mg}\mathfrak{D}'+(4\alpha-\beta+1)\mathfrak{l}\right)$$

$$+\cos(4\eta-2v-u)\left(-\frac{27e}{16mg}\mathfrak{G}'+\frac{27e}{16mg}\mathfrak{J}'+(4\alpha-2\beta-1)\mathfrak{l}\right)$$

$$+\cos(4\eta-2v+u)\left(-\frac{27e}{16mg}\mathfrak{G}'+\frac{27e}{16mg}\mathfrak{J}'+(4\alpha-2\beta+1)\mathfrak{l}\right)$$

$$+\cos(4\eta-u)\left(+\frac{9e}{16mg}\mathfrak{G}'+(4\alpha-1)\mathfrak{l}\right)$$

$$+\cos(4\eta+u)\left(+\frac{9e}{16mg}\mathfrak{G}'+(4\alpha+1)\mathfrak{l}\right)$$

$$+\cos(2\eta-4v-u)\left(-\frac{9e}{8mg}\mathfrak{J}'+(2\alpha-4\beta\ \ 1)\mathfrak{l}\right)$$

$$+\cos(2\eta-4v+u)\left(-\frac{9e}{8mg}\mathfrak{J}'+(2\alpha-4\beta+1)\mathfrak{l}\right)$$

$$+\cos(4\eta-4v-u)\left(-\frac{81e}{3mg}\mathfrak{J}'+(4\alpha-4\beta-1)\mathfrak{l}\right)$$

$$+\cos(4\eta-4v+u)\left(-\frac{81e}{8mg}\mathfrak{J}'+(4\alpha-4\beta+1)\mathfrak{l}\right)$$

LIX.

Collectis hinc valoribus coefficientium assumtorum, obtinebitur longitudo lunae vt sequitur:

$\varphi =$

$$\Phi = C + 13{,}3682\,\omega$$

$$\begin{array}{llll}
- 22728'' \sin v & + 2823'' \sin(2\eta - 3v) & + 100'' \sin u \\
- 1081 \sin 2v & + 47 \sin(2\eta + 3v) & - 23 \sin(v - u) \\
- 128 \sin 3v & - 246 \sin(4\eta - v) & - 20 \sin(v + u) \\
- 700 \sin 2\eta & + 41 \sin(4\eta + v) & + 22 \sin(2v - u) \\
- 3594 \sin(2\eta - 2v) & + 379 \sin(4\eta - 3v) & + 21 \sin(2v + u) \\
+ 306 \sin(2\eta + 2v) & - 2 \sin(4\eta + 3v) & + 2 \sin(3v - u) \\
- 3762 \sin(2\eta - v) & & + 2 \sin(3v + u) \\
- 115 \sin(2\eta + v) & & - 2 \sin(4v - u) \\
 & & - 2 \sin(4v + u)
\end{array}$$

$$\begin{array}{ll}
+ 17'' \sin(2\eta - u) & - 1' \sin(2\eta - 4v - u) \\
+ 19 \sin(2\eta + u) & - 1 \sin(2\eta - 4v + u) \\
+ 1 \sin(4\eta - u) & - 11 \sin(4\eta - 2v - u) \\
+ 1 \sin(4\eta + u) & - 10 \sin(4\eta - 2v + u) \\
+ 6 \sin(2\eta - v - u) & - 28 \sin(4\eta - 3v - u) \\
+ 5 \sin(2\eta - v + v) & - 23 \sin(4\eta - 3v + u) \\
+ 60 \sin(2\eta - 2v - u) & - 98 \sin(4\eta - 4v - u) \\
- 194 \sin(2\eta - 2v + u) & - 245 \sin(4\eta - 4v + u) \\
- 6 \sin(2\eta + 2v - u) & + 2 \sin(4\eta - v - u) \\
- 6 \sin(2\eta + 2v + u) & + 2 \sin(4\eta - v + u) \\
+ 6 \sin(2\eta - 3v - u) & \\
+ 8 \sin(2\eta - 3v + u) &
\end{array}$$

LX.

Plurimae igitur prodeunt inaequalitates ab excentricitate solis pendentes, quarum nonnullae ita sunt magnae,

gnae, vt sine notabili errore omitti nequeant; cuiusmodi sunt imprimis, quae ab angulis $2\eta - 2v \mp u$ et $4\eta - 4v \mp u$ pendent. Sed in his fere idem incommodum vsu venit, quo methodus praecedens premebatur, quod magnitudo harum inaequalitatum per Theoriam non satis accurate definiri queat. Cum enim pro his terminis inueniendis diuisores $2\alpha - 2\beta \mp 1$ et $4\alpha - 4\beta \mp 1$ fiant perquam exigui, manifestum est in diuidendis terminos minimos neglectos non exigui fore momenti: praecipue cum pro litteris $\mathfrak{g}'$ et $\mathfrak{h}'$ termini maiores fere se mutuo destruxissent. Vnde cum ex valore Φ tantum termini maiores $\mathfrak{B}'$, $\mathfrak{D}'$, $\mathfrak{G}'$ et $\mathfrak{J}'$ essent adhibiti, perspicuum est si etiam reliqui minores fuissent introducti, ex iis insignem mutationem in valore coefficientium $\mathfrak{g}'$ et $\mathfrak{h}'$ orituram fuisse.

LXI.

De his autem inaequalitatibus tenendum est, eas per satis notabile temporis spatium vix immutari; nam inaequalitates ab angulo $2\eta - 2v + u$, ob quantitatem $2\alpha - 2\beta + 1 = -0{,}1594$ periodum habent annorum circiter $6\frac{1}{4}$ annorum, et interuallo 19 annorum ter tantum reuoluuntur: et inaequalitas ab angulo $4\eta - 4v + u$ pendens spatio 29 annorum 19 periodos absoluit. Ex quo cum istae inaequalitates per theoriam saltem propemodum fuerint definitae, eas deinceps per obseruationes accuratius definiri conueniet: nisi forte quis laborem in se suscipere voluerit, calculum hic adumbratum multo accuratius instituendi terminorumque hic omissorum rationem habendi; tum vero etiam valores ξ, q et $\Phi - v$

multo

multo maiori ſtudio, quam hic feci, euolui oporteret, quoniam in horum determinatione multa neglexi, quae in calculo tandem ad notabilem quantitatem excreſcere potuiſſent.

LXII.

Interim tamen hic notari conuenit, hac methodo eas tantum inaequalitates prodire incertas, quae ſatis longis periodis abſoluuntur; quae incertitudo minus officit, cum per obſeruationes facilius emendari poſſit: praecedente vero methodo etiam aliae inaequalitates minoribus periodis circumſcriptae aliquantum incertae prodierunt, quod ſane ingens erat incommodum. Vnde ex hac parte haec methodus poſterior priori anteferenda videtur: verum ſi ingentem inaequalitatum numerum ſpectemus, quibus non ſolum lunae longitudo afficitur, ſed etiam longitudo apogei, calculus tantopere fit operoſus, vt etiamſi has formulas accuratiſſime euoluerem, tamen in praxi difficillimi foret vſus. Quin etiam plurimae inaequalitates in motum apogei ingredi videntur, quarum effectum deinceps per alias longitudinis inaequalitates iterum deſtrui oportet, ita vt ſatius fuiſſet illas penitus omittere.

LXIII.

Multitudo autem harum inaequalitatum, quibus tam apogei, quam ipſius lunae longitudo turbatur, inde potiſſimum originem trahit, quod inaequalitates excentricitatis prae eius quantitate media admodum ſint notabiles, atque adeo quadrantem mediae quantitatis ſuperent; ita vt prae ea negligi minime queant. Multo plures autem

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tem adhuc inaequalitates eſſent acceſſurae, ſi excentricitas lunae media adhuc eſſet minor, quo certe caſu calculi difficultates inſuperabiles euaſiſſent: hoc vero ipſo caſu methodus prior multo tractabilior redderetur, cum enim pleraeque inaequalitates ibi multo minores prodirent. Atque ob hanc cauſam minus expedire videtur, anomaliam lunae ita conſtituere, vt eius ſinus tam pro maximis quam pro minimis diſtantiis lunae a terra plane euaneſcat, etiamſi haec ratio naturae rei maxime conſentanea videatur.

LXIV.

Cum igitur numerus inaequalitatum iam tantopere increuerit, facile perſpicitur eum adhuc multo magis auctum iri, ſi eas inaequalitates, quae cum a parallaxi ſolis, tum ab eius inclinatione ad eclipticam eſſem euoluturus, quo labore propterea, cum eius vſus fere nullus futurus eſſet, ſuperſedebo. Interim tamen hinc tantum colligere licet, inaequalitates ab angulis $2\eta - 2v \mp u$ et $4\eta - 4v \pm u$ ortas, minime eſſe contemnendas; quae cum methodo praecedente ſint vel omiſſae vel non ſatis accurate determinatae, ſine dubio cauſam in ſe continent, quod etiam accuratiſſimae tabulae per obſeruationes emendatae adhuc vltra $4'$ ſaepe a veritate aberrent.

LXV.

Sufficiat igitur methodum expoſuiſſe, cuius ope inaequalitates lunae tam ratione apogei, quam longitudinis ac latitudinis verae ex anomalia hic adhibita determinari queant; neque propterea laborem calculi reliquaquarum inaequalitatum, quae vel ex ſolis parallaxi vel ex inclinatione orbitae lunaris ad eclipticam oriuntur, ſuſcipio; quippe quarum numerus, ſiquidem omnes, quae alicuius momenti eſſent futurae, perſequi vellem, in immenſum excreſceret. Non ſolum autem multitudo inaequalitatum hanc methodum omni vtilitate in praxi priuabit, ſed etiam ingentes aequationes, quas determinatio apogei, atque anomaliae inde pendentis requirit, ita ſunt comparatae, vt ipſae iam ſatis exactam tam longitudinis quam anomaliae cognitionem requirant; quae res etſi initio ſupponi poſſent, deinceps iterata eadem operatione accuratius definiendae, tamen quia correctio apogei vltra 30° gradus aſſurgere poteſt, calculus ob inaequalitatum multitudinem per ſe taediosus, nimis crebro repeti deberet, antequam de concluſione certi eſſe poſſemus.

APPLICATIO FORMULARUM INUENTARUM AD ALIOS CALCULOS LUNARES.

LXVI.

Cum igitur calculus inaequalitatum motus lunae hactenus duplici modo ſit inſtitutus, dum priori anomalia vera regulis Keplerianis conformis eſt aſſumta, poſteriori vero ita conſtituta, vt eius ſinus tam pro maximis lunae a terra diſtantiis quam pro minimis prorſus euaneſceret, quorum vterque vti vidimus incommodis non careret: ita etiam infinitis aliis modis lunae inaequalitates repraeſentari poterunt, quos breuiter expoſuiſſe haud abs

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re fore arbitror. Nullum enim eſt dubium, quin inter hos infinitos modos quidam reperiantur, qui ipſi naturae rei magis ſint conſentanei, neque iis incommodis laborent, quibus vtrumque expoſitum non mediocriter impediri comperimus; etiamſi adhuc difficillimum videatur, inter hanc infinitam multitudinem modum conuenientiſſimum eligere.

LXVII.

Poſtquam autem inueſtigationem ab aequationibus differentio-differentialibus ad aequationes ſimpliciter differentiales produximus, etiamſi ad hoc anomalia vera v cuius ſinus in maximis ac minimis lunae a terra diſtantiis euaneſcat, ſimus vſi, tamen haec conditio iam iterum exui poteſt. Cum enim tam ſinus quam coſinus ipſius v vbique per quantitatem q ſit multiplicatus, loco harum duarum variabilium q et v iam alias duas variabiles in calculum introducere poterimus, quod commodiſſime fiet ponendo $q \cos v = r$ et $q \sin v = s$, vt ſit $qq = rr + ss$ et $\tang v = \frac{s}{r}$, tum enim vi formularum §. IX. exhibitarum habebimus iſtas aequationes:

$$dr = -s\,d\Phi + \frac{2M}{A}\,d\omega \sqrt{Ap};$$

$$ds = r\,d\Phi + \left(\frac{N}{A} - \frac{Ms}{A(1-r)}\right) d\omega \sqrt{Ap}.$$

 LXVIII.

LXVIII.

Hinc autem porro erit: $x = \frac{p}{1-r}$;

$$dp = -\frac{2Mp\,d\omega}{A(1-r)} \sqrt{Ap} \quad \text{et} \quad d\Phi = \frac{d\omega(1-r)^2}{pp} \sqrt{Ap}, \text{ tum}$$

vero vt ante $du = d\theta = \frac{d\omega(1-e\cos u)^2}{(1-ee)\sqrt{(1-ee)}}$.

Deinde vero si statuamus $p = b(1+\xi)$, reliquasque denominationes in §§. 11, 12, 13. factas adhibeamus, obtinebimus:

$$\frac{M}{A}\sqrt{Ap} = \frac{3(1+3ee)}{2m}\frac{(1-e\cos u)^3}{1-r}(1+\tfrac{3}{2}\xi)\sin 2\eta$$
$$+\frac{3n}{8m}\cdot\frac{(1-e\cos u)^4}{(1-r)^2}(1+\tfrac{4}{3}\xi)(\sin\eta + 5\sin 3\eta)$$

$$\frac{N}{A}\sqrt{Ap} = -\frac{(1+3ee)}{2m}\cdot\frac{(1-e\cos u)^3}{1-r}(1+\tfrac{3}{2}\xi)(1+3\cos 2\eta)$$
$$-\frac{3n}{8m}\cdot\frac{(1-e\cos u)^4}{(1-r)^2}(1+\tfrac{4}{3}\xi)(3\cos\eta + 5\cos 3\eta)$$
$$+m(1-r)^2(1-\tfrac{3}{2}\xi)\,\Pi - i$$

atque

$$x = \frac{b(1+\xi)}{1-r}\;;\quad d\xi = -\frac{2(1+\xi)\,d\omega}{1-r}\cdot\frac{M}{A}\sqrt{Ap};$$

ac tandem:

$$d\Phi = m\,d\omega(1-\tfrac{3}{2}\xi + \tfrac{15}{8}\xi\xi)(1-r)^2$$

LXIX.

Hinc iam omnium differentialium rationes ad $d\omega$ habentur, erit enim

$$\frac{d\xi}{d\omega} = -\frac{2(1+\xi)}{1-r}\cdot\frac{M}{A}\sqrt{Ap}:$$

$$\frac{d\Phi}{d\omega} = m(1-\tfrac{3}{2}\xi + \tfrac{15}{8}\xi\xi)(1-r)^2$$

$$\frac{du}{d\omega} = \frac{d\theta}{d\omega} = \frac{(1-e\cos u)^2}{(1-ee)\sqrt{(1-ee)}} \quad \text{et} \quad \frac{d\eta}{d\omega} = \frac{d\Phi - d\theta}{d\omega}.$$

$$\frac{dr}{d\omega} = -ms(1-\tfrac{3}{2}\xi + \tfrac{15}{8}\xi\xi)(1-r)^2 + 2.\frac{M}{A}\sqrt{Ap}$$

$$\frac{ds}{d\omega} = mr(1-\tfrac{3}{2}\xi + \tfrac{15}{8}\xi\xi)(1-r)^2 + \frac{N}{A}\sqrt{Ap} - \frac{s}{1-r}\cdot\frac{M}{A}\sqrt{Ap}$$

$$\frac{d\pi}{d\omega} = -\frac{1}{m}\sin(\theta-\pi)\sin(\Phi-\pi)\left(\frac{3(1+3ee)(1+\tfrac{3}{2}\xi)}{(1-r)^3}\cos\eta\right.$$
$$\left.+\tfrac{1}{4}n(3+5\cos 2\eta)\right)$$

$$\frac{d\,l\,tg\,\rho}{d\omega} = -\frac{1}{m}\sin(\theta-\pi)\cos(\Phi-\pi)\left(\frac{3(1+3ee)(1+\tfrac{3}{2}\xi)}{(1-r)^3}\cos\eta\right.$$
$$\left.+\tfrac{1}{4}n(3+5\cos 2\eta)\right)$$

at est $\Pi = 1 - \lambda - \frac{3}{4}\operatorname{tang}\rho^2 - \frac{3}{4}\operatorname{tang}\rho^2\cos 2(\Phi-\pi)$, vbi pro λ assumi potest $1 - \frac{3}{4}\operatorname{tang} \epsilon^2$, denotante ϵ inclinationem mediam orbitae lunaris ad eclipticam, vt sit $\Pi = \frac{3}{4}\operatorname{tang}\epsilon^2 - \frac{3}{4}\operatorname{tang}\rho^2\,[1+\cos 2(\Phi-\pi)]$

LXX.

Quodſi iam pro r ſtatueretur iſte valor $k \operatorname{cof} v$, ita vt k eſſet quantitas conſtans, orirетur modus initio traditus inaequalitates lunae repraeſentandi, foret enim tum v anomalia vera Kepleri et k denotaret excentricitatem orbitae lunaris. Vnde patet etiam inaequalitates lunae per methodum primam erutas, ex his formulis inueniri poſſe, neque ad hoc aequationibus ſecundi gradus eſſe opus. Reduceretur autem hoc caſu indoles differentio-differentialium ad inuentionem quantitatis s, quem in finem pro s aſſumi deberet ſeries quaedam ſinuum angulorum η, v, u et $\Phi - \pi$ formatorum cum indefinitis coefficientibus, quos deinceps determinare liceret: hoc autem modo ſolutio primum tradita eſſet proditura.

LXXI.

Cum ſit $p = b(1+\xi)$ et $\sqrt{A} = m\sqrt{b^3}$, ob $dx = -\frac{sd\omega}{p}\sqrt{Ap}$, fiat $\frac{dx}{d\omega} = -\frac{mbs}{\sqrt{(1+\xi)}} = -mbs(1-\frac{1}{2}\xi+\frac{3}{8}\xi\xi)$, vnde patet ſi eiusmodi anomalia v introducatur, vt ſit $s = q \operatorname{fin} v$, ſiue q ſit quantitas conſtans ſiue variabilis, tum hanc anomaliam tam in maximis quam minimis diſtantiis ſinum euaneſcentem eſſe habituram. Ac ſi pro q quantitas vel conſtans vel ex angulis cognitis compoſita aſſumatur, tum inde coefficientes aſſumti ac praeterea valor litterae r determinabitur. Sin autem pro q eiusmodi quantitas incognita aſſumatur, vt ſit praeterea $r = q \operatorname{cof} v$, tum ſolutio ante expoſita reſultabit.

LXXII.

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XII.

LXXII.

Semper autem vſus aſtronomicus exigit, vt anomalia vera quaedam angulo v contenta introducatur, id quod infinitis modis fieri poteſt. Quo autem quantitas r variabilitatem diſtantiarum lunae a terra accuratius exprimat, et valor ipſius ξ quam minimas mutationes ſubeat, neceſſe eſt, vt quantitas r huiusmodi contineat terminum $k \operatorname{cof} v$, vbi k excentricitatem deſignet, qui ſit quaſi eius pars praecipua; hocque etiam locum habet, ſi pro r ſumatur $q \operatorname{cof} v$, denotante q quantitatem variabilem, quippe cuius pars potior excentricitatem k praebere debet. Verum praeterea quantitas r alios terminos continere poteſt, qui ab angulo v vel pendeant vel non pendeant: ita poni poſſet: $r = k \operatorname{cof} v + A \operatorname{cof} 2\eta + B \operatorname{cof} 4\eta + C \operatorname{cof}(2\eta - v)$ etc. quo valore aſſumto litterae quoque s, ξ cum reliquis ſuos valores debitos obtinerent.

LXXIII.

Hoc modo illud incommodum euitari poteſt, quo methodum in hoc additamento traditam laborare vidimus, ſi excentricitas orbitae lunaris eſſet nimis parua, vel adeo euaneſcens; tum enim diſtantiae maximae et minimae non amplius ab anomalia penderent, ſed potius ab angulo η, atque imprimis quidem a coſinu dupli anguli 2η. Caſu ergo quo excentricitas plane euaneſcit, pro variabili r, cuius loco vtique noua variabilis introduci debet, non conueniet anomaliam v introducere, ſed praeſtabit aſſumi ſeriem coſinuum ex ſolis angulis 2η, u

 et

et $\Phi-\pi$ conſtantem, quorum coefficientes etſi ſunt cónſtantes, tamen quia terminorum numerus in infinitum excurrit, vicem novae variabilis ſuſtinebunt. Tum autem valor ipſius s ex ſimili ſerie ſinuum eorundem angulorum conſtabit.

LXXIV.

Quodſi ergo rem generatim pro quacunque excentricitate expedire velimus, poterimus ad hos terminos, qui ex hypotheſi excentricitatis euaneſcentis prodeunt, adhuc adiungere terminos ex anomalia v formatos. Ita neglectis tam inaequalitatibus parallacticis, quam iis quae tum ab excentricitate orbitae ſolaris, tum ab inclinatione orbitae lunaris ad eclipticam pendent, poni conueniet:

$$r = k\,\text{cof}\,v + A\,\text{cof}\,2\eta + B\,\text{cof}(2\eta-v) + C\,\text{cof}(2\eta+v) + D\,\text{cof}\,4\eta + E\,\text{cof}(4\eta-v)\ \text{etc.}$$

$$s = \Delta k\,\text{fin}\,v + \mathfrak{A}\,\text{fin}\,2\eta + \mathfrak{B}\,\text{fin}(2\eta-v) + \mathfrak{C}\,\text{fin}(2\eta+v) + \mathfrak{D}\,\text{fin}\,4\eta + \mathfrak{E}\,\text{fin}(4\eta-v)\ \text{etc.}$$

$$\xi = ..\,\text{cof}\,v + ..\,\text{cof}\,2\eta + ..\,\text{cof}(2\eta-v) + ..\,\text{cof}(2\eta+v) + ..\,\text{cof}\,4\eta + ..\,\text{cof}(4\eta-v)\ \text{etc.}$$

Atque ſi hoc modo omnes angulorum 2η et v combinationes adhibeantur, hique valores in aequationibus ſupra datis ſubſtituantur, primo inde elicietur ratio $dv : d\omega$, ac deinceps coefficientes determinationes ſuas nanciſcentur.

LXXV.

Manebunt autem coefficientes vnius ſeriei veluti ipſius r indeterminati, propterea quod ipſa ſeries haec ab

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ab arbitrio noſtro pendet, dum pro r vel ſolum primum terminum k cof v, vel quotquot lubuerit, aſſumere potuiſſemus. Hinc autem id commodi conſequemur, vt iſtos coefficientes ad ſcopum quam conuenientiſſime definire valeamus: ſcilicet eos ita definiri conueniet, vt primo nullius reliquorum coefficientium determinatio lubrica et incerta euadat, vti in vtraque methodo expoſita vſu venit: deinde vero vt nulli coefficientes fiant nimis magni praeter neceſſitatem, ita vt eorum effectus per alios terminos iterum deſtrui neceſſe ſit. Fateri quidem cogor calculum hoc modo inſtituendum admodum futurum eſſe prolixum, verum fortaſſe in ipſa operatione non contemnenda ſe offerent compendia; vnde confido hanc ſpeculationem, etiamſi mihi ipſi eam ſuſcipere non vacet, vſu non eſſe carituram.

BEROLINI, EX OFFICINA MICHAELIS.

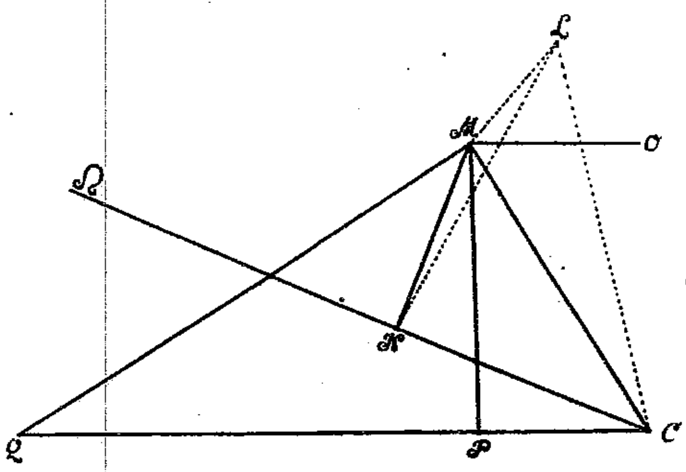

Fig. 1.

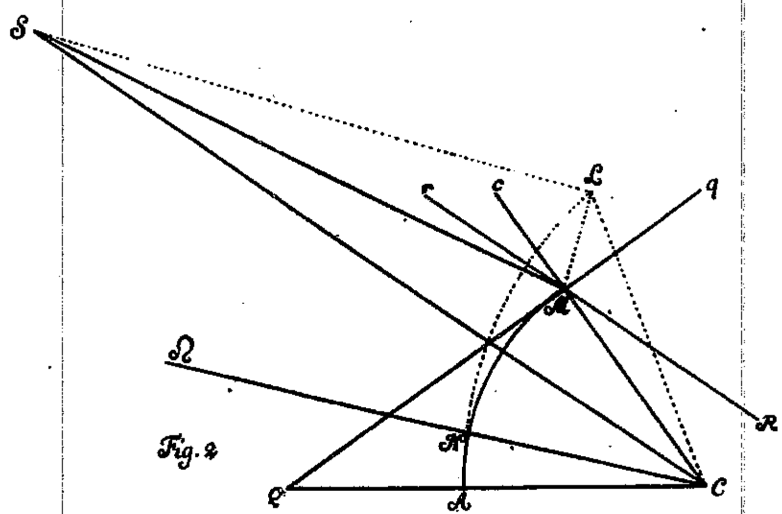

Fig. 2