Geometrica et Spherica Quaedam

Mémoires de l'Académie des sciences de St-Pétersbrug **5** (1812), p.96-114.

1. Not long ago, as I was considering the situation in which, in a triangle ABC, line segments Aa, Bb, and Cc are drawn arbitrarily from each vertex to its opposite side so as to coincide in a point O, the following question occurred to me: how might one be able, given the two parts of each of these lines segments, to construct the triangle itself? I soon noticed that this question does not admit of a solution in general, unless a certain definite relation holds between the six parts. Accordingly I happened upon the following rather memorable result:

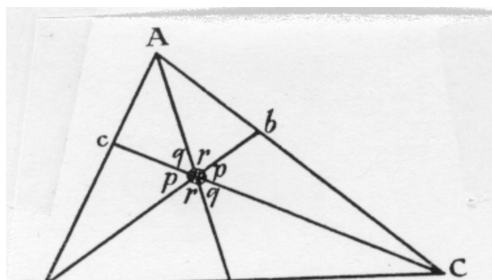

Fig. 1

Theorem

If, in a triangle ABC, line segments Aa, Bb, and Cc are drawn arbitrarily from each vertex to its opposite side so as to coincide in a point O, then the following property always holds, namely:

$$\frac{AO}{Oa}\cdot\frac{BO}{Ob}\cdot\frac{CO}{Oc}=\frac{AO}{Oa}+\frac{BO}{Ob}+\frac{CO}{Oc}+2.$$

Proof

2. To demonstrate this, let us name the parts described above:

$$\begin{aligned} AO &= A,\ BO=B, CO=C,\\ Oa &= a, Ob=b, OC=c, \end{aligned}$$

and let us also name all six angles formed around the point O as they have been marked in the figure, where it is immediately evident that $p+q+r=180°$. Consequently, from the usual formula in which from two sides of a triangle and the included angle its area is determined, we will have the area

$$AOc=\frac{1}{2}Ac\sin q,$$

and the area

$$BOc = \frac{1}{2} Bc \sin p,$$

and also the area

$$AOB = \frac{1}{2} AB \sin (p+q) .$$

Moreover, $\sin(p+q) = \sin r$, whence, since the last triangle is the sum of the previous ones, the following equation is deduced:

$$AB \sin r = Ac \sin q + Bc \sin p.$$

Similarly, the remaining parts give these equations:

$$\begin{aligned} BC \sin p &= Ba \sin c + Ca \sin q, \\ CA \sin q &= Cb \sin p + Ab \sin r, \end{aligned}$$

3. So that we might fit these equations more nearly to our use, let us transform them into the following:

$$\begin{aligned} \frac{\sin r}{c} &= \frac{\sin q}{B} + \frac{\sin p}{A}, \\ \frac{\sin p}{a} &= \frac{\sin r}{C} + \frac{\sin q}{B}, \\ \frac{\sin q}{b} &= \frac{\sin p}{A} + \frac{\sin r}{C}. \end{aligned}$$

Hence it is evident that these three angles p, q, and r can be related to the terms A, B and C or to the terms a, b and c—whichever one prefers.

4. Further, if we put $A = \alpha a$, $B = \beta b$, and $C = \gamma c$, we would also have:

$$\begin{aligned} \frac{\sin p}{A} &= \frac{\sin p}{\alpha a} = P, \\ \frac{\sin q}{B} &= \frac{\sin q}{\beta b} = Q, \\ \frac{\sin r}{C} &= \frac{\sin r}{\gamma c} = R, \end{aligned}$$

In this way we will obtain the following three very simple formulas:

$$\gamma R = P + Q, \quad \alpha P = Q + R, \quad \beta Q = R + P.$$

Moreover the differences of these equations at once supply ratios between pairs of the terms P, Q, and R; thereupon we arrive at the following proportions:

$$\begin{aligned} P &: R = \gamma + 1 : \alpha + 1, \\ Q &: P = \alpha + 1 : \beta + 1, \\ R &: Q = \beta + 1 : \gamma + 1, \end{aligned}$$

whence this double proportion clearly follows:

$$P : Q : R = \frac{1}{\alpha+1} : \frac{1}{\beta+1} : \frac{1}{\gamma+1}.$$

In the same way it is evident that the terms P, Q, and R are related to α, β, and γ respectively.

5. Moreover, since the first of our three equations provides

$$R = \frac{P+Q}{\gamma},$$

and from the second one gets $R = \alpha P - Q$, these two values, when set equal, will produce the following ratio between P and Q: $\frac{P}{Q} = \frac{\gamma+1}{\alpha\gamma-1}$. Wherefore, since $\frac{P}{Q} = \frac{\beta+1}{\alpha+1}$, we will arrive in this way at an equation free of the terms P, Q, and R, that is: $\alpha\beta\gamma = \alpha + \beta + \gamma + 2$, which manifestly is the property in the stated theorem, since

$$\alpha = \frac{AO}{Oa}, \quad \beta = \frac{BO}{Ob}, \quad \gamma = \frac{CO}{Oc}.$$

On the basis of this theorem let us approach the question mentioned in the beginning.

Problem

Given that, in a triangle ABC three line segments Aa, Bb and Cc are drawn from each vertex to the opposite side so as to coincide in a point O, and the two parts of each of these three lines are named as

$$\begin{aligned} AO &= A, \quad BO = B, \quad CO = C, \\ Oa &= a, \quad Ob = b, \quad Oc = c, \end{aligned}$$

and assuming that the property, previously described, holds: to investigate the construction of the triangle itself from the six given quantities.

Solution

6. Let all notation remain as in the demonstration of the theorem, namely: $A = \alpha a$, $B = \beta b$, $C = \gamma c$, also $\sin p = \alpha a P$, $\sin q = \beta b Q$, $\sin r = \gamma c R$; above all let us keep in mind that the following relationship must hold between the terms α, β, and γ : $\alpha\beta\gamma = \alpha + \beta + \gamma + 2$.

7. Moreover, since the terms P, Q and R stand in the same ratio among themselves as do the fractions $\frac{1}{\alpha+1}$, $\frac{1}{\beta+1}$, and $\frac{1}{\gamma+1}$, let us set

$$P = \frac{\Delta}{\alpha+1}, \quad Q = \frac{\Delta}{\beta+1}, \quad R = \frac{\Delta}{\gamma+1},$$

so that in this way the terms P, Q and R, and therefore the sines of the inner angles p, q, and r, are banished from the calculation, in place of which a single unknown Δ enters. Once its value is found, everything that is required for the construction of the triangle will be made known.

8. It would also be helpful to investigate the unknown Δ using the condition that the sum of the three angles $p+q+r$ makes two right angles, which is to say that $\sin r = \sin(p+q)$. Moreover, as we have so far found expressions only for the sines of these angles, it will be fitting to reduce our equation to sines alone. So that this may be more easily accomplished, let us set $\sin p = f$, $\sin q = g$, and $\sin r = h$, so that necessarily

$$h = f\sqrt{1-gg} + g\sqrt{1-ff},$$

from which we should remove the radicals.

Hence let squares be taken and we will have

$$hh = ff + gg - 2ffgg + 2fg\sqrt{(1-ff)(1-gg)}.$$

Let the terms without radicals be transferred to the left, and, once we have taken squares again, we arrive at the following equation:

$$f^4 + g^4 + h^4 - 2ffgg - 2ffhh - 2gghh + 4ffgghh = 0.$$

9. Therefore, since $f = \sin p$, through the notation before, the outcome is

$$f = \alpha a P = \frac{\alpha a \Delta}{\alpha+1}, \quad g = \frac{\beta b \Delta}{\beta+1}, \quad h = \frac{\gamma c \Delta}{\gamma+1}.$$

Accordingly let us put, for the sake of brevity, $\frac{\alpha a}{\alpha+1} = F$, $\frac{\beta b}{\beta+1} = G$, and $\frac{\gamma c}{\gamma+1} = H$, so that $f = F\Delta$, $g = G\Delta$, and $h = H\Delta$, and these values, being substituted in the equation recently found, will produce an equation divisible by Δ^4, which becomes:

$$F^4 + G^4 + H^4 - 2F^2G^2 - 2F^2H^2 - 2G^2H^2 + 4F^2G^2H^2\Delta^2 = 0,$$

whence we conclude

$$\Delta^2 = \frac{2F^2G^2 + 2F^2H^2 + 2G^2H^2 - F^4 - G^4 - H^4}{4F^2G^2H^2},$$

so that our unknown Δ is now perfectly determined, being

$$\Delta = \frac{\sqrt{2F^2G^2 + 2F^2H^2 + 2G^2H^2 - F^4 - G^4 - H^4}}{2FGH},$$

which can also be expressed through its factors in this way:

$$\Delta = \frac{\sqrt{(F+G+H)(F+G-H)(F+H-G)(G+H-F)}}{2FGH}.$$

10. Since, therefore, the terms F, G, and H are given immediately from the known quantities A, B, C and a, b, c, the expression just found, which can seem not a little involved, nevertheless can be constructed quite easily using the area of a triangle. For let a triangle be constructed, the sides of which are F, G, and H, and let is area, which we may call M^2, be sought. It is seen to be:[1]

$$M^2 = \frac{1}{4}\sqrt{(F+G+H)(F+G-H)(F+H-G)(G+H-F)}.$$

Using this notation gives $\Delta = \frac{2M^2}{FGH}$, and with this value known the whole affair is brought to a conclusion, for from it we find at once the sines of the angles p, q, and r, to wit:

$$\sin p = \frac{\alpha a}{\alpha+1}\Delta, \quad \sin q = \frac{\beta b}{\beta+1}\Delta, \quad \sin r = \frac{\gamma c}{\gamma+1}\Delta.$$

Moreover, when just one of these angles has been found, the triangle itself can be constructed immediately—this is understood easily in itself.

11. In addition, once I had accomplished these things, I noticed that the theorem given above can be stated much more conveniently, and more elegantly, in the following way:

Theorem

If in a certain triangle ABC line segments Aa, Bb and Cc are drawn from the angles A, B, and C so as to coincide in a point O, then the following property will always hold:

$$\frac{Oa}{Aa} + \frac{Ob}{Bb} + \frac{Oc}{Cc} = 1.$$

In other words, if the three segment parts Oa, Ob, and Oc are divided one-by-one by their corresponding whole segments, the three resulting fractions, when added together, will equal unity.

Proof (Using the Previous Result)

12. Setting, as before, $AO = \alpha \cdot Oa$, $BO = \beta \cdot Ob$, and $CO = \gamma \cdot OC$, we have demonstrated previously that $\alpha\beta\gamma = \alpha + \beta + \gamma + 2$. Let the following expression be added to both sides:

$$\alpha\beta + \alpha\gamma + \beta\gamma + \alpha + \beta + \gamma + 1,$$

[1]Euler is thinking of Heron's formula for the area of a triangle. Letting the sides of a triangle be a, b, c and defining the *semiperimeter* as $s = \frac{a+b+c}{2}$, Heron's formula gives the area as $\sqrt{s(s-a)(s-b)(s-c)}$.

and on the left side we will get

$$(\alpha+1)(\beta+1)(\gamma+1)$$

while on the right we will get

$$\alpha\beta+\alpha\gamma+\beta\gamma+2(\alpha+\beta+\gamma)+3,$$

and this last formula clearly factors into three parts:

$$(\alpha+1)(\beta+1)+(\alpha+1)(\gamma+1)+(\beta+1)(\gamma+1).$$

Having substituted this form, let the equation be divided on each side by the product

$$(\alpha+1)(\beta+1)(\gamma+1)$$

and the result will be

$$1=\frac{1}{\gamma+1}+\frac{1}{\beta+1}+\frac{1}{\alpha+1}. \qquad \text{Q.E.D.}$$

13. From here it is also possible to derive the following noteworthy property:

$$\frac{\alpha}{a+1}+\frac{\beta}{\beta+1}+\frac{\gamma}{\gamma+1}=2,$$

for if the former equation is added to this one, the following identity arises: $1+1+1=3$.

A Very Simple Proof

Depending on Common Elements

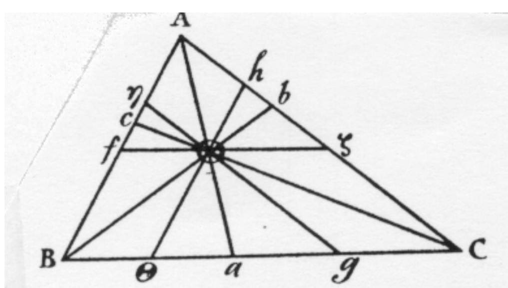

Fig. 2

14. Through the point O (Fig. 2) let line segments be drawn to each side of the triangle, so that $f\zeta$ is parallel to BC, $g\eta$ is parallel to AC and $h\theta$ is parallel to AB. At once it is evident that

$$\frac{Bf}{AB}+\frac{A\eta}{AB}+\frac{f\eta}{AB}=1.$$

Now, on account of the similar triangles ABa and AfO we will have

$$: Bf : BA = Oa : Aa,$$

and thus the first fraction $\frac{Bf}{AB}$ turns out to be $\frac{Oa}{Aa}$. Then, since $\Delta BAb \sim \Delta B\eta O$, we will have $A\eta : AB = Ob : Bb$, whence $\frac{A\eta}{AB}$ becomes $\frac{Ob}{Bb}$. Finally,

$$\Delta fO\eta \sim \Delta BCA, \text{ whence } f\eta : BA = fO : BC,$$

from which the fraction $\frac{f\eta}{AB}$ becomes $\frac{fO}{BC}$. Now $fO = B\theta$, so, since triangle $BCc \sim \Delta\theta CO$, the fraction $\frac{B\theta}{BC}$ will be $\frac{Oc}{Cc}$, and, these values having been substituted, the identity $\frac{Bf+A\eta+f\eta}{AB} = 1$ assumes the form

$$\frac{Oa}{Aa} + \frac{Ob}{Bb} + \frac{Oc}{Cc} = 1,$$

which is the very equality that was to be demonstrated.

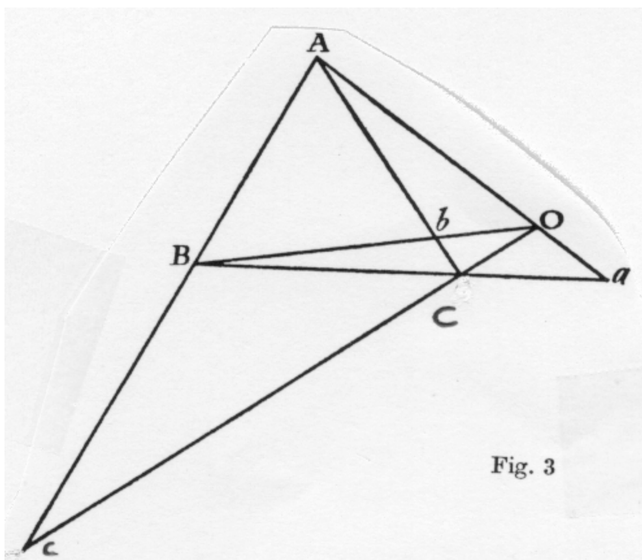

Fig. 3

15. This remarkable property also holds when the point O is taken anywhere outside the triangle, as in Figure 3[2], provided that notation along the sides Aa, Bb and Cc is correctly established.[3] So in this figure, for the segment Aa

[2] The *Opera Omnia* edition had the point labels C and c reversed, which did not seem correct to me.

[3] Euler's argument in this section, if indeed any argument is being given, is quite unclear to me. His concluding equation is correct, however. The general result, where O may be located anywhere, even outside the triangle, may be stated as follows:

$$\frac{Oa}{Aa} + \frac{Ob}{Bb} + \frac{Oc}{Cc} = 1,$$

where any term on the left-hand side is to be taken as negative if the numerator and denominator, considered as directed line segments, point in opposite directions. Thus, in the case Euler considers in Figure 3, Ob and Bb have opposite directions, and so $\frac{Ob}{Bb}$ appears as negative in Euler's conclusion.

we will set $AO = A$ and $Oa = a$, but as regards the segment Bb, having set $BO = B$, we will let $Ob = -b$, and for segment Cc we ought to put $CO = C$ and $Oc = -c$. From this we will have

$$Aa = A + a, \quad Bb = B + b, \quad Cc = -(C + c).$$

Therefore, since we always have

$$\frac{a}{a+A} + \frac{b}{b+B} + \frac{c}{c+C} = 1,$$

for the lines drawn in the figure we will have

$$\frac{Oa}{Aa} - \frac{Ob}{Bb} + \frac{Oc}{Cc} = 1.$$

16. Moreover, once this property has been established, the area of the whole triangle ABC may be found rather conveniently. For since the area of triangle $AOB = \frac{1}{2}AB \sin r$ and $\sin r = CR$ (see section 4), then the area of AOB is $\frac{1}{2}ABC \cdot R$. Similarly the area of AOC will be found to be $\frac{1}{2}ABC \cdot Q$ and the area of BOC will be $\frac{1}{2}ABC \cdot P$, and thus the total area of the triangle will be equal to

$$\frac{1}{2}ABC\,(P + Q + R)\,.$$

17. Further we put

$$P = \frac{F\Delta}{A}, \quad Q = \frac{G\Delta}{B}, \quad R = \frac{H\Delta}{C}.$$

Moreover we had $F = \frac{A}{\alpha+1}$, $G = \frac{B}{\beta+1}$, and $H = \frac{C}{\gamma+1}$ (see section 9); the area of the triangle will become

$$ABC\Delta\left(\frac{1}{\alpha+1} + \frac{1}{\beta+1} + \frac{1}{\gamma+1}\right).$$

Moreover we have shown that

$$\frac{1}{\alpha+1} + \frac{1}{\beta+1} + \frac{1}{\gamma+1} = 1,$$

so the area of our triangle will be $\frac{1}{2}ABC\Delta$. Furthermore we considered the triangle formed by the three sides F, G, and H, and we set its area equal to M^2, having found which we obtained the value $\Delta = \frac{2M^2}{FGH}$. One this value is substituted, the area of our triangle will be expressed as follows: $\frac{ABCM^2}{FGH}$. Therefore, since

$$F = \frac{A}{\alpha+1}, G = \frac{B}{\beta+1}, \text{ and } H = \frac{C}{\gamma+1},$$

the area will be

$$(\alpha + 1)\,(\beta + 1)\,(\gamma + 1)\,M^2$$

and thus the area of the original triangle ABC and the area, M^2, of the triangle introduced to help us, have a rather simple ratio, namely:

$$(\alpha + 1)\,(\beta + 1)\,(\gamma + 1) : 1.$$

Alternatively, using line segments, the area ratio is:

$$ABC : M^2 = Aa \cdot Bb \cdot Cc : Oa \cdot Ob \cdot Oc.$$

Spherical Triangles

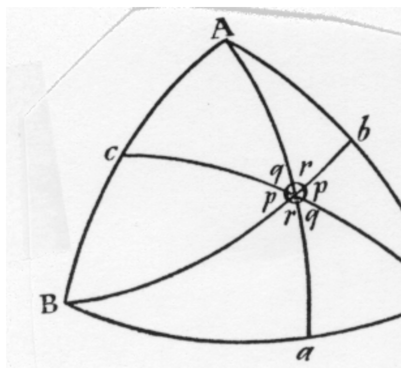

Fig. 4

18. The results we have so far found concerning triangles in the plane can be modified for spherical triangles. That is, let a spherical triangle ABC (Figure 4) be set forth, in which arcs from each vertex are drawn to the opposite sides so as to coincide at a point O, and we are to investigate what relation should hold between between the parts of these arcs, so that, if they are given, the triangle itself may be constructed. To this end the following theorem should be treated first:

Theorem

If, in a spherical triangle ABC arcs Aa, Bb, and Cc are drawn from each vertex to its opposite side so as to coincide in a point O, and if for the sake of brevity we set

$$\frac{\tan AO}{\tan Oa} = \alpha, \quad \frac{\tan BO}{\tan Ob} = \beta, \quad \frac{\tan CO}{\tan Oc} = \gamma,$$

then we will always have $\alpha\beta\gamma = \alpha + \beta + \gamma + 2$, which may be reduced to

$$\frac{1}{\alpha+1} + \frac{1}{\beta+1} + \frac{1}{\gamma+1} = 1.$$

Proof

19. Let the angles around the point of intersection be named as marked in the figure, and also set arcs $AO = A$, $BO = B$, $CO = C$, $Oa = a$, $Ob = b$, $Oc = c$. In triangle AOc we will have

$$\tan AcO = \frac{\sin A \sin q}{\cos A \sin c - \sin A \cos c \cos q},$$

and in triangle BOc we will have

$$\tan BcO = \frac{\sin B \sin p}{\cos B \sin c - \sin B \cos c \cos p}.$$

20. Now since these two angles. when summed together, make two right angles, the sum of their tangents should equal zero, whence the following equation arises:

$$\sin A \cos B \sin c \sin q - \sin A \sin B \cos c \cos p \sin q + \sin B \cos A \sin c \sin p - \sin A \sin B \cos c \cos q \sin p = 0$$

which reduces to the following simpler equation:

$$\sin A \cos B \sin c \sin q + \sin B \cos A \sin c \sin p = \sin A \sin B \cos c \sin r,$$

from which we gather

$$\sin r = \frac{\sin A \cos C \sin c \sin q + \sin B \cos A \sin c \sin p}{\sin A \sin B \cos c},$$

which in turn produces the following equation:

$$\frac{\sin r}{\tan c} = \frac{\sin p}{\tan A} + \frac{\sin q}{\tan B}.$$

In the same way we will have:

$$\begin{aligned}\frac{\sin p}{\tan a} &= \frac{\sin q}{\tan B} + \frac{\sin r}{\tan C},\\ \frac{\sin q}{\tan b} &= \frac{\sin r}{\tan C} + \frac{\sin p}{\tan A}.\end{aligned}$$

21. Moreover, since in the theorem we set

$$\frac{\tan A}{\tan a} = \alpha, \quad \frac{\tan B}{\tan b} = \beta, \quad \frac{\tan C}{\tan c} = \gamma,$$

when these values are substituted the three equations assume the following form:

$$\begin{aligned}\frac{\sin r}{\tan c} &= \frac{\sin p}{\alpha \tan a} + \frac{\sin q}{\beta \tan b},\\ \frac{\sin p}{\tan a} &= \frac{\sin q}{\beta \tan b} + \frac{\sin r}{\gamma \tan c},\\ \frac{\sin q}{\tan b} &= \frac{\sin r}{\gamma \tan c} + \frac{\sin p}{\alpha \tan a}.\end{aligned}$$

22. Let us now further set

$$\frac{\sin p}{\alpha \tan a} = P, \quad \frac{\sin q}{\beta \tan b} = Q, \quad \frac{\sin r}{\gamma \tan c} = R.$$

Once this is done, our three equations will be

$$\gamma R = P + Q, \quad \alpha P = Q + R, \quad \beta Q = R + P.$$

The first of these becomes $R = \frac{P+Q}{\gamma}$, and the second becomes $R = \alpha P - Q$. When these values are set equal they give $\frac{P}{Q} = \frac{\gamma+1}{\alpha\gamma-1}$. Then the second equation, when the third is subtracted from it, furnishes $\alpha P - \beta Q = Q - P$, whence one deduces that $\frac{P}{Q} = \frac{\beta+1}{\alpha+1}$, so that we have the following equation:

$$\frac{\gamma+1}{\alpha\gamma-1} = \frac{\beta+1}{\alpha+1},$$

which, when unravelled and reduced to order, is gathered up as:

$$\alpha\beta\gamma = \alpha + \beta + \gamma + 2.$$

Q.E.D.

Another Proof
Drawn From the First Elements of Geometry

23. Let us imagine a plane tangent to the sphere at the point O. We will not represent it by a figure, as it is easily understood. Let straight lines be drawn from the center of the sphere through the points A, B, C, and a, b, c, meeting the plane in the points A', B', C', a', b', c'. If from O the lines $OA', OB', OC', Oa', Ob', Oc'$ are drawn in the plane, then it is evident that we will have $OA' = \tan A$, and similarly $OB' = \tan B$, and $OC' = \tan C$, and also $Oa' = \tan a$, $Ob' = \tan b$, and $Oc' = \tan c$, and also the same angles, given by the letters p, q, and r, will be present in the plane.

24. In this way we have now obtained a plane triangle $A'B'C'$, from each of whose vertices straight lines $A'a', B'b', C'c'$ have been drawn to the opposite sides so as to coincide at the point O, and so the whole business has been reduced to the case of a plane triangle, wherefore, if we set

$$\frac{OA'}{Oa'} = \frac{\tan A}{\tan a} = \alpha, \quad \frac{OB'}{Ob'} = \frac{\tan B}{\tan b} = \beta, \quad \frac{OC'}{Oc'} = \frac{\tan C}{\tan c} = \gamma,$$

we will certainly have, as was proved before, both

$$\alpha\beta\gamma = \alpha + \beta + \gamma + 2$$

and

$$\frac{1}{\alpha+1} + \frac{1}{\beta+1} + \frac{1}{\gamma+1} = 1.$$

Problem

Given that, for a spherical triangle ABC in which line segments Aa, Bb, Cc are drawn from each vertex to the opposite side so as to coincide at a point O, the six parts of these arcs

$$\begin{aligned} AO &= A, \quad BO = B, \quad CO = C, \\ Oa &= a, \quad Ob = b, \quad Oc = c, \end{aligned}$$

are known: to find the triangle itself from these six given quantities (with the relationship, previously demonstrated, holding between them).

Solution

25. Let us set[4], as has been done already in the theorem:

$$\frac{\sin p}{\alpha \tan a} = P, \quad \frac{\sin q}{\beta \tan b} = Q, \quad \frac{\sin r}{\gamma \tan c} = R,$$

[4] The remarks of sections 25 and 26 essentially repeat the argument of sections 19-22. It is unclear to me why Euler feels he must again derive the property $\alpha\beta\gamma = \alpha + \beta + \gamma + 2$, as this relation is explicitly assumed to hold in the construction problem he has set for himself.

and we obtain the following three equations (section 21): $\gamma R = P + Q$, $\alpha P = Q + R$, $\beta Q = R + P$, from whose differences we deduce at once the following formulas:

$$\begin{aligned} \gamma R - \alpha P &= P - R, \text{ whence it follows that } \frac{P}{R} = \frac{\gamma+1}{\alpha+1}. \\ \text{Further, } \alpha P - \beta Q &= Q - P, \text{ whence we get } \frac{Q}{P} = \frac{\alpha+1}{\beta+1}. \\ \text{Finally, } \gamma R - \beta Q &= Q - R, \text{ from which we get } \frac{R}{Q} = \frac{\beta+1}{\gamma+1}; \end{aligned}$$

whence it is clear that the terms P, Q, R have the same ratio among themselves as do the fraction:

$$\frac{1}{\alpha+1}, \frac{1}{\beta+1}, \frac{1}{\gamma+1},$$

wherefore we may set:

$$P = \frac{\Delta}{\alpha+1}, \quad Q = \frac{\Delta}{\beta+1}, \quad R = \frac{\Delta}{\gamma+1}.$$

26. Now from the first equation we deduce $R = \frac{P+Q}{\gamma}$, from the second $R = \alpha P - Q$, and these being set equal, we get $\frac{P}{Q} = \frac{\gamma+1}{\alpha\gamma-1}$. In the same way as before, we find that $\frac{P}{Q} = \frac{\beta+1}{\alpha+1}$, which two values, if they are set equal, produce the same condition that was already proved in the theorem.

27. Moreover, with these values being used in place of the terms $P, Q,$, and R, we thence produce the following values for the sines of the angles p, q, and r:

$$\sin p = \tan A \cdot \frac{\Delta}{\alpha+1}, \quad \sin q = \tan B \cdot \frac{\Delta}{\beta+1}, \quad \sin r = \tan C \cdot \frac{\Delta}{\gamma+1.}$$

Now as we further set, for the sake of brevity

$$\frac{\tan A}{\alpha+1} = F, \quad \frac{\tan B}{\beta+1} = G, \quad \frac{\tan C}{\gamma+1} = H,$$

the condition, that the sum of the angles $p + q + r$ is 180°, straightforwardly supplies for us, as in the preceding problem, the following value for:

$$\Delta = \frac{\sqrt{(F+G+H)(F+G-H)(F+H-G)(G+H-F)}}{2FGH}.$$

Whence if we imagine another triangle, whose sides are F, G, H and whose area we call M^2, we will straightforwardly have, as before, that $\Delta = \frac{2M^2}{FGH}$, and thus the term Δ is determined by just these known quantities.

28. Therefore since the value of Δ itself is known, we will have

$$\sin p = \frac{2M^2}{GH}, \quad \sin q = \frac{2M^2}{FH}, \quad \sin r = \frac{2M^2}{FG},$$

whence each angle p, q, r may be determined; moreover for the construction of the triangle it suffices to know just one of them. It will also help to note that the formula we have found provides two values for each angle, each of which is complementary to the other, so that two solutions always obtain; and since the cosines of obtuse angles are negative, the negative solution ought to be set so as to make $\cos r = -\cos(p+q)$. Therefore in this way the spherical problem is solved completely.

Supplement

Containing a very simple analysis both for the demonstration of the theorem and for the solution of the problem previously posed.

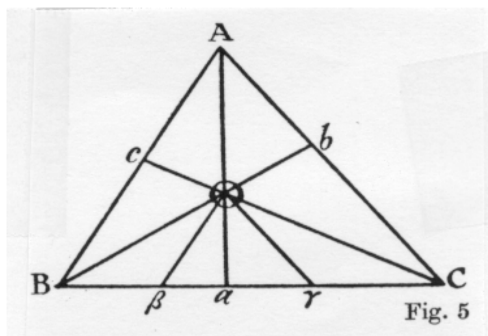

Fig. 5

29. Let ABC be a triangle (Figure 5) from whose angles straight lines Aa, Bb, Cc have been drawn to the opposite sides so as to coincide at a point O, and let us set $AO = A$, $BO = B$, $CO = C$, $Oa = a$, $Ob = b$, $Oc = c$, and from the point O let lines $O\beta$ and $O\gamma$ be drawn parallel to the sides AB and AC. These lines will suffice to carry through the whole business.

30. Once these lines have been established, the similarity of triangles BCb and $B\gamma O$ will give $\frac{Cy}{BC} = \frac{Ob}{Bb}$. Further, the similarity of triangles CBc and $C\beta O$ will give $\frac{B\beta}{BC} = \frac{Oc}{Cc}$. Finally, triangle $\beta O\gamma$ is similar to triangle BAC and in each the lines Oa and Aa are drawn similarly, whence $\frac{\beta\gamma}{BC} = \frac{Oa}{Aa}$, and thus we have these three equations:

$$\begin{aligned} \frac{Oa}{Aa} &= \frac{a}{A+a} = \frac{\beta\gamma}{BC}, \\ \frac{Ob}{Bb} &= \frac{b}{B+b} = \frac{C\gamma}{BC}, \\ \frac{Oc}{Cc} &= \frac{c}{C+c} = \frac{B\beta}{BC}. \end{aligned}$$

Therefore the sum of the three fractions will be:

$$\frac{\beta\gamma + C\gamma + B\beta}{BC} = 1,$$

which is without doubt a very brief demonstration of the theorem previously brought to light by circuitous wanderings.

31. Let us now put, for the sake of brevity,

$$\frac{Oa}{Aa} = \frac{a}{A+a} = \alpha, \quad \frac{Ob}{Bb} = \frac{b}{B+b} = \beta, \quad \frac{Oc}{Cc} = \frac{c}{C+c} = \gamma,$$

where the terms α, β, γ should not be confused with those utilized earlier. From this we will have

$$a = \frac{\alpha A}{1-\alpha} \quad \text{or} \quad A = \frac{a\,(1-\alpha)}{\alpha},$$

and in the same way

$$b = \frac{\beta B}{1-\beta} \quad \text{or} \quad B = \frac{b\,(1-\beta)}{\beta},$$

and

$$c = \frac{\gamma C}{1-\gamma} \quad \text{or} \quad C = \frac{c\,(1-\gamma)}{\gamma}.$$

Also now the terms α, β, γ are now related thus: $\alpha + \beta + \gamma = 1$.

32. Let us now denote by x the entire base BC of the triangle, and its three parts $B\beta, \beta\gamma, \gamma C$ will be determined thus:

$$\beta\gamma = \alpha x, \quad B\beta = \gamma x, \quad C\gamma = \beta x.$$

Then, since triangle $Oa\beta$ is similar to triangle AaB, we have $\frac{Ba}{B\beta} = \frac{Aa}{AO}$, whence we deduce that the segment

$$Ba = \frac{B\beta \cdot Aa}{AO} = \frac{\gamma x\,(A+a)}{A} = \frac{\gamma x}{1-\alpha}.$$

In the same way we get:

$$Ca = \frac{\beta x}{1-\alpha}.$$

33. Now that these values have been found, let us consider only the triangle BOC (Figure 6), in which the line Oa has been drawn, and, as we have established, let $OB = B$, $OC = C$, and $Oa = a$. The base BC is cut by this last line at a, so that

$$Ba : Ca = \gamma : \beta,$$

and thus we have been reduced to the following problem: given the triangle sides OB and OC and segment Oa and the ratio in which the base BC is cut, to construct the triangle itself.

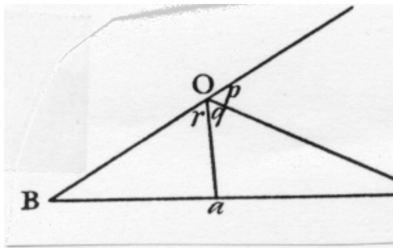

Fig. 6

34. To this end let us denote the angles terminating at the point O as they have been marked in the figure: their sum is equal to two right angles. For the purpose of discovering the ratio of the sines of these angles, the triangle OBa gives the following proportion:

$$\sin B : \sin r = Oa : Ba = a : \frac{\gamma x}{1-\alpha}.$$

But the entire triangle BOC provides $\sin p : \sin B = BC : OC$. When these proportions are multiplied together we get:

$$\sin p : \sin r = a : \frac{C\gamma}{1-\alpha},$$

so that

$$\sin r = \frac{C\gamma \sin p}{a\,(1-\alpha)},$$

and similarly one obtains

$$\sin q = \frac{B\beta \sin p}{a\,(1-\alpha)}.$$

Thus the ratio between the sines of the angles q and r and the sine of the angle p is determined, since we are assuming that the quantities A, B, C and a, b, c, (and hence also α, β, γ) are known.

35. If we consider more closely the three angles p, q, r, whose sum is equal to two right angles, another triangle is given, whose angles are equal to these three. Since the three sines, that is $\sin p$, $\sin q$, $\sin r$, have the same ratio among themselves as do the do the three known quantities $a\,(1-\alpha)$, $B\beta$, $C\gamma$, if we construct a triangle from these three sides, the angles opposite each side will be the very angles p, q, r we have sought.

36. The whole business will therefore be finished if we construct (Figure 7[5]) a triangle $O\mathtt{CB}$, whose base $\mathtt{BC}$ is $a\,(1-\alpha) = Aa$ and whose sides are $\mathtt{B}O = B\beta$ and $\mathtt{C}O = C\gamma$, and whose angles will also be: to vertex O, p; to vertex $\mathtt{B}$, r; and to vertex $\mathtt{C}$, q. If we then extend the side $\mathtt{B}O$ to a point B, so that $OB = B$, and if in the other side $O\mathtt{C}$ is divided at a point C so that $OC = C$, then these two lines already make between them an angle $q + r$, as the previous computation

[5] The points B and C were not given in this figure, but they form the base of the lower triangle.

asserts; next, if we draw from O a line Oa parallel to BC, we will have a point a in the side BC, and for that reason Oa will be equal to a, and in this way our problem is thoroughly solved. For if the segment BO is extended to b to make the other part Ob, and CO to c and aO to A, then the figure initially proposed will be complete, and in this way from the six given quantities A, B, C along with a, b, c related so that the property we found is satisfied, the entire triangle is constructed by this exceedingly simple operation.

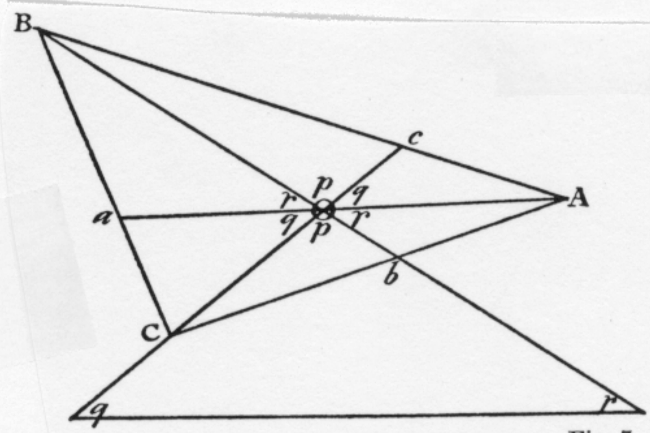

Fig. 7

37. The situation, which we have examined here, ought to be considered worthy of greater attention, for in the beginning it appeared to require rather abstruse and burdensome computations, even though, once all the difficulties were overcome, we were brought round to a very simple and elegant solution.