Imaginary numbers are real

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Math 17: Beyond Calculus
Dartmouth College
Spring 2013
(2) MWF 1:45 - 2:50; Th 1:00–1:50

Contents

1 Introduction

This course is intended as an introduction to math beyond calculus. The idea is to introduce math that is fun, challenging, and important, and prepare and inspire you to major in math. The primary target audience is first-year students who have completed Math 11, 12, or 13. A motivated student coming from Math 8 should be able to do fine. At the same time, the couse should still be challenging to students coming out of Math 22 or 24. (There will also be some upperclass students in the class, and I hope they will enjoy it, but the class is primarily aimed at first-years.)

This is a topics course, with no set material that we have to go through. This spring, the focus will be on complex numbers. Complex numbers are deeply implicated in the structure of our universe, and they play a central role in many areas of mathematics, as well as in physics and engineering (and music!). It would be good for students to come to know and love them at the start of their college careers. This will be a primary goal of the course.

We'll be discussing a lot of other topics as well: There are plenty of other things things students ought to come to know and love (cardinal and ordinal numbers; quadratic reciprocity; Goedel's theorem; the halting problem; . . .). Some topics will be more accessible than others, and I don't expect all students to follow everything. There won't be any formal exams, though there will be quizzes on things that I think every student should master. There will be frequent written assignments, though most will not resemble a typical 'problem set'. A big emphasis will be on computer explorations, and major independent projects, which will demand a lot of work.

An important component of the course will be learning basic Mathematica programming, and the mathematical typesetting language Latex.

2 Course texts

These books are meant as resources. We won't read any of them cover to cover (except for the last, which students at whom this course is aimed are unlikely to be able to put down, once they get into it). We'll refer to them by the indicated nicknames.

Tale *An Imaginary Tale: The Story of the Square Root of Minus One*
Paul J. Nahin
ISBN-10: 0691146004 — ISBN-13: 978-0691146003
Amazon price: 11.53
http://www.amazon.com/Imaginary-Tale-Princeton-Science-Library/dp/0691146004

What *What Is Mathematics? An Elementary Approach to Ideas and Methods*
Courant, Robbins, and Stewart
ISBN-10: 0195105192 — ISBN-13: 978-0195105193
Amazon price: 14.25
http://www.amazon.com/Mathematics-Elementary-Approach-Ideas-Methods/dp/019510519

Road *The Road to Reality: A Complete Guide to the Laws of the Universe*
Roger Penrose
ISBN-10: 0679776311 — ISBN-13: 978-0679776314
Amazon price: 15.75
http://www.amazon.com/Road-Reality-Complete-Guide-Universe/dp/0679776311

Age *The Age of Entanglement: When Quantum Physics Was Reborn*
Louisa Gilder

ISBN-10: 1400095263 — ISBN-13: 978-1400095261
Amazon price: 11.40
http://www.amazon.com/Age-Entanglement-Quantum-Physics-Vintage/dp/1400095263

3 Mathematica

Download Mathematica here:
http://caligari.dartmouth.edu/downloads/mathematica/
Note the link to the quick intro videos:
http://www.wolfram.com/broadcast/screencasts/handsonstart/

To use Mathematica, you need to be on the internal Dartmouth network, either via an ethernet connection or WiFi to Dartmouth Secure (*not* Dartmouth Public). From off campus, you can use VPN:
http://www.dartmouth.edu/comp/internet/offcampus/vpn/juniper.html

4 LaTeX

The easiest way to use LaTeX is via LyX:
http://www.lyx.org/
Download here:
http://www.lyx.org/Download
Note that for Mac OS X you will need to install MacTeX:
http://www.tug.org/mactex

An alternative front end for LaTeX is the TeXworks interface. This comes with MacTeX. On Windows download this:
http://www.tug.org/texlive/windows.html

5 Organization

Instructor

Peter Doyle, 331 Kemeny Hall. Instead of office hours I'll be scheduling help sessions. Blitz me any time to ask a question or set up a meeting.

Class meetings

The class meets in the 2 slot, MWF 1:45–2:50. We will be using the X-hour, Th 1:00-1:50. Keep this time open!

When you will not able to attend class, I would appreciate it if you would send me email in advance.

Components of the course

The major components of the course are assignments; occasional quizzes; and two projects.

Assignments Assignments will be due at the beginning of class on specified due dates. Extensions will be granted in case of illness, and in case of other pressing conflicts provided you ask in advance. Late assignments will be marked off.

Quizzes The aim of the quizzes will be to make sure you have mastered basic material and techniques. The main focus of the course will be the assignments and projects.

Projects The two projects will give you a chance to do an investigation on your own. Projects need not involve a Mathematica component, though I expect most projects will have one. The first project will be due, in both printed and electronic form, at the end of the seventh week of the term (Friday 10 May). The second project will be due, in printed, electronic, and poster form, at 4:00pm on Tuesday 28 May, when students will present their projects at the Math Department student poster session running 4–6 pm. The second project may be either an extension of the first project, or an entirely new project. You may work together in pairs on these projects if you choose.

Grading

Grades will be subjective, based on my assessment of what students have put into and gotten out of the course. As a starting point, I will compute an average of scaled scores on assignments, quizzes, and projects.

Honor Code

Students are encouraged to work together to do homework problems. What is important is a student's eventual understanding of homework problems, and not how that is achieved. The honor principle applies to homework in the following way. What a student turns in as a written homework solution is to be his or her own understanding of how to do the problem. Students must state what sources they have consulted, with whom they have collaborated, and from whom (other than the instructor) they have received help. Students are discouraged from using solutions to problems that may be posted on the web, and as just stated, must reference them if they use them. The solutions you submit must be written by you alone. Any copying (electronic or otherwise) of another person's solutions, in whole or in part, is a violation of the Honor Code.

On projects, no copying of text, computer code, or graphics will be permitted without prior written permission from me.

If you have any questions as to whether some action would be acceptable under the Academic Honor Code, please speak to me, and I will be glad to help clarify things. It is always easier to ask beforehand than to have trouble later!

Disabilities

I encourage any students with disabilities, including "invisible" disabilities such as chronic diseases and learning disabilities, to discuss appropriate accommodations with me, which might help you with this class, either after class or during office hours. Dartmouth College has an active program to help students with disabilities, and I am happy to do whatever I can to help out, as appropriate.

Sources

- The sections on 'honor code' and 'disabilities' are adapted from Pete Winkler's syllabus for Math 100, Winter 2010.

6 Monday 25 March

Remember to bring your laptop along to class on Wednesday! For Wednesday:

1. Download and install Mathematica, and explore the 'quick intro' videos. (See above.)

2. Download LyX (see above) and explore the tutorial.

3. Read [*What*, 1Supp.1.1 and 1.Supp.4.1–3 (pp. 21–25 and 42–48)], on the fundamental theorem of arithmetic and the Euclidean algorithm. This will explain what we did in class today. On Wednesday we'll explore this further using Mathematica.

4. Read [*Road*, Ch. 4 (pp. 71–85)]. This will serve as an introduction to complex numbers. You may want to compare with the treatments in [*What*, 2.5.1–3 (pp. 88–103)] and [*Tale*, 3.1 (pp. 48–60)].

7 Thursday 28 March

At the X-hour today we worked on basic Mathematica programming, specifically recursive programming, starting with summing the entries in a list, reversing a list, and building up to writing our own gcd program to duplicate the functionality of Mathematica's builtin GCD program. I plan to go over this at the beginning of class tomorrow. In preparation, please look over the following Mathematica notebook and make sure you understand what is going on:
http://www.math.dartmouth.edu/ doyle/docs/17.2013/GCD.nb

Here's a pdf version:
http://www.math.dartmouth.edu/ doyle/docs/17.2013/GCD.pdf

8 Monday 1 April

Notebook showing powers of a complex number:
http://www.math.dartmouth.edu/ doyle/docs/17.2013/pow.nb
Notebook showing recursive definitions of plus, times, and power:
http://www.math.dartmouth.edu/ doyle/docs/17.2013/peano.nb

8.1 Assignment to work on for Friday 5 April

Investigate the following using Mathematica.

1. Plot the 100th-degree Taylor approximation to the sine function. How far out does the agreement with sine extend? Explain.

2. Draw the tesselation illustrating the proof of the Pythagorean theorem, as in Figure 2.3 of *Road*. (Don't worry about the labels or how far the tesselation extends.)

3. Using recursion, write functions to:

   - Find the length of a list.
   - Add the integers in a list.
   - Multiply the integers in a list.
   - Tell if a given integer occurs in a list.
   - Replace all occurrences of a with b in a given list.
   - Count the open braces in a nested list.
   - Generate the list representing a given integer n, as on page 64 of *Road*, namely $0 = \{\}, 1 = \{0\} = \{\{\}\}, 2 = \{0, 1\} = \{\{\}, \{\{\}\}\}, \ldots$.

4. What fraction of the integers from 1 to 10^k are square-free for $k = 1, \ldots, 6$? What is the limiting density of square-free integers? What about cube-free integers?

5. Plot the Gaussian primes (some of them).

6. Rewrite the gcd function so as to use subtraction rather than Mod.

7. Rewrite the gcd function to return (x, y) such that $xa + yb = \gcd(a, b)$.

8. Rewrite the gcd function to work with Gaussian integers, and return x, y such that $xa + yb = \gcd(a, b)$.

9. Write a function to factor a given rational prime p into Gaussian primes.

9 Wednesday 3 April: Assignment for Monday 8 April

Prepare to turn in solutions to problems 1–5 above on Monday 8 April. On Thursday 4 April we'll meet in the X-hour to continue work on these problems.

There will be no class on Friday 5 April.

10 Why i?

Let z be a function of t, and suppose that $z(t) \neq 0$, so we can write

$$z' = az.$$

To see how fast $|z|$ is changing, we compute

$$\frac{d}{dz}|z|^2 = (z\bar{z})' = z'\bar{z} + z\bar{z}' = z'\bar{z} + z\bar{z'} = (a + \bar{a})z\bar{z} = 2\Re(a)|z|^2.$$

So, to keep $|z|$ constant, we must make a purely imaginary.

If $z(t) = \exp(it)$ we have $z' = iz$ so $a = i$, and since $z(0) = 1$, $\exp(it)$ remains on the unit circle for all t. This goes a long way toward explaining Euler's formula

$$\exp(it) = \cos t + i \sin t.$$

Now let's look at the Schroedinger equation ...