EXPLICIT FORMULAS FOR STRANGENESS OF PLANE CURVES

ALEXANDER SHUMAKOVITCH

Abstract. Recently V. I. Arnold [2] introduced three numerical characteristics of generic immersion of the circle into the plane. These characteristics were defined axiomatically, making hard to calculate them and to evaluate their range. In this paper we prove explicit formulas for one of these characteristics called strangeness, making its calculation much easier. We also find sharp upper and lower bounds for the range of strangeness and, in particular, prove all conjectures formulated by Arnold [2].

Introduction

In this paper by a *plane curve* or simply a *curve* we mean an (C^1-smooth) immersion of the circle S^1 into the plane $\mathbb{R}^2$. We say that a curve is *generic* if it has neither self-intersection points with multiplicity greater than 2 nor self-tangency points, and at each double point its branches are transversal to each other.

In the paper [2] V. I. Arnold has shown that it is possible to assign for each generic curve three numerical characteristics which are invariant under homotopy in the class of generic curves. They are denoted by St, J^+, and J^-. The notation St originates from the name *strangeness* which was chosen by Arnold.

The invariants St, J^+, and J^- in some sense characterize, respectively, the number of triple point, direct and reverse self-tangency perestroikas which are needed to be in a generic regular homotopy connecting one curve with another one (the strict definitions are in 1.3). All three invariants were defined axiomatically using recurrence relations, making their calculations quite hard for curves with many self-intersection points.

Recently O. Ya. Viro [7] found the following explicit formulas for $J^+(C)$ and $J^-(C)$ in terms of topological properties of the pair $(\mathbb{R}^2, C)$:

$$J^+(C) = 1 - \sum_X \operatorname{ind}^2_{\tilde C}(X)\,\chi(X) + n,$$
$$J^-(C) = 1 - \sum_X \operatorname{ind}^2_{\tilde C}(X)\,\chi(X),$$

where $\tilde C$ is a family of circles obtained as a result of smoothing of the curve C at each double point with respect to orientation (see Figure 1); X runs through the collection of the components of the set $\mathbb{R}^2 \setminus \tilde C$, $\operatorname{ind}_{\tilde C}(X)$ is the index of points of the component X with respect to $\tilde C$, χ is the Euler characteristic, and n is the number of double points of the curve C.

Key words and phrases. Immersion of the circle into the plane, generic immersion, Whitney index, regular homotopy, perestroikas of a plane curve, Arnold's invariants of a plane curve, strangeness.

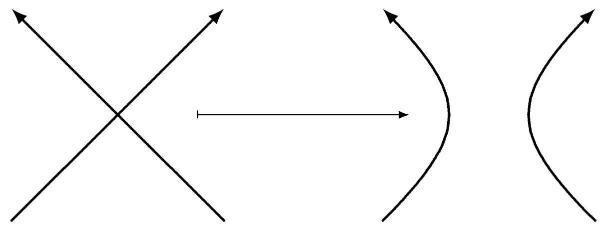


Figure 1. The curve smoothing at a self-intersection point.

In this paper we prove some formulas for $\mathrm{St}(C)$. All of them, actually, are modifications of the same formula represented in different terms. Besides the main formulas (which are written in a similar form to the Viro's formulas for $J^{\pm}$), there are also reformulations in terms of the Gauss word of a curve C (see, for instance, [3]) and ascending knot diagrams.

In section 1 we formulate the main results, which are proved in section 2. In section 3 we analyze the value range of St and prove all Arnold's conjectures [2] about St. In the Appendix there are proofs of some auxiliary facts.

The author would like to express gratitude to O. Ya. Viro for his valuable discussions.

1. The main definitions and formulations

1.1. Whitney index. By *Whitney index* or simply *index* of a curve we mean the total rotation number of the tangent vector to the curve obtained when we move along the oriented curve (it is obvious that this is the degree of the map which associate a direction of the tangent vector to every point of the circle). We denote the index of a curve C by $\mathrm{ind}(C)$. It is easy to see that the index does not change under a *regular homotopy* of a curve that is a C^1-smooth homotopy in the class of C^1-immersions.

Let us remark that under the change of orientation (that is an orientation reversing reparametrization) the index changes its sign. Moreover, its sign changes under the change of orientation of the plane $\mathbb{R}^2$. Therefore in the case of a nonoriented curve (or plane) we can define only the absolute value of index. The simplest examples of the curve with indices $0, \pm 1, \pm 2, \dots$ are shown in Figure 2.

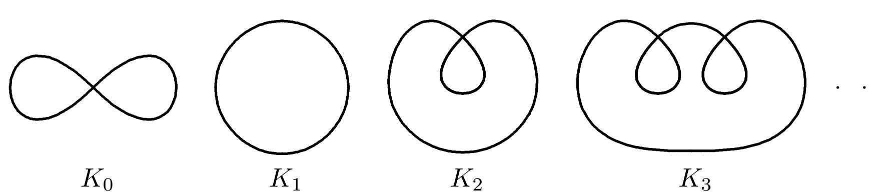


Figure 2. The standard curves with indices $0, \pm 1, \pm 2, \pm 3, \dots$

1.1.A. Theorem (Whitney [8]). *Two curves C_1 and C_2 can be transformed into each other by a regular homotopy if and only if* $\mathrm{ind}(C_1) = \mathrm{ind}(C_2)$.

1.2. Generic homotopies and perestroikas. Let us call a nongeneric curve as a *first order singular curve* or simply as a 1-*singular curve* if it differs from a generic curve either in exactly one point of triple transversal self-intersection or in exactly one point of self-tangency. In the case of generic regular homotopy between two generic curves C_1 and C_2 we can meet only finitely many nongeneric curves and each of them is a 1-singular curve.

In a point of self-tangency the velocity vectors of the tangent branches can have either the same direction or the opposite one. In the first case the self-tangency is called *direct* and in the second one called *reverse*. Let us remark that the type of the self-tangency point does not change under reversing of orientation.

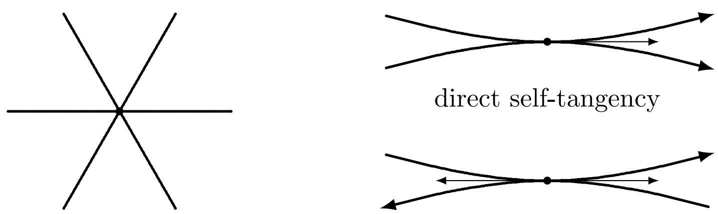


FIGURE 3. The types of 1-singular curves.

Hence we have three types of 1-singular curves (see Figure 3). The passages through singular curves during a generic homotopy correspond to three *perestroikas* of a curve (see Figure 4).

Let us consider the triple point perestroika more carefully. Just before and just after the passage through a 1-singular curve with triple point, there is a small triangle close to the place of perestroika which is formed by three curve branches. This triangle is called *vanishing*. The orientation of the curve defines the cyclic order of move along the edges of the vanishing triangle. This cyclic order gives us the triangle orientation and, therefore, the orientation of its edges. Let us denote by q the number of edges of the vanishing triangle for which obtained orientation coincides with the curve orientation (it is obvious that q takes value between 0 and 3).

Let us define a *sign* of the vanishing triangle as $(-1)^q$. Remark that the sign does not change under a reverse of curve orientation. Some examples of vanishing triangles with different signs are shown in Figure 5. It is demonstrated in the same figure that before and after the perestroika the vanishing triangle signs are different.

1.2.A. DEFINITIONS (ARNOLD [2]). 1. The triple point perestroika is called *positive* if we have a positive vanishing triangle after it. 2. The self-tangency perestroika is called *positive* if it increase (by 2) the number of self-intersection points of the curve.

1.3. Three Arnold's invariants. The following theorem represents the definition of generic curve invariants which was promised above.

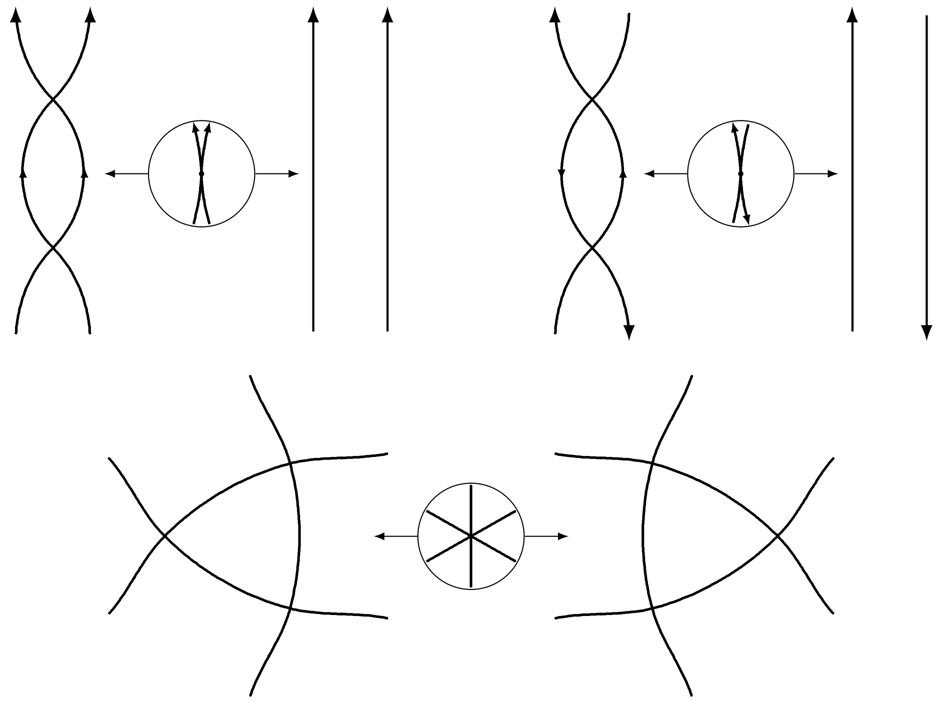


Figure 4. The perestroikas of generic curves.

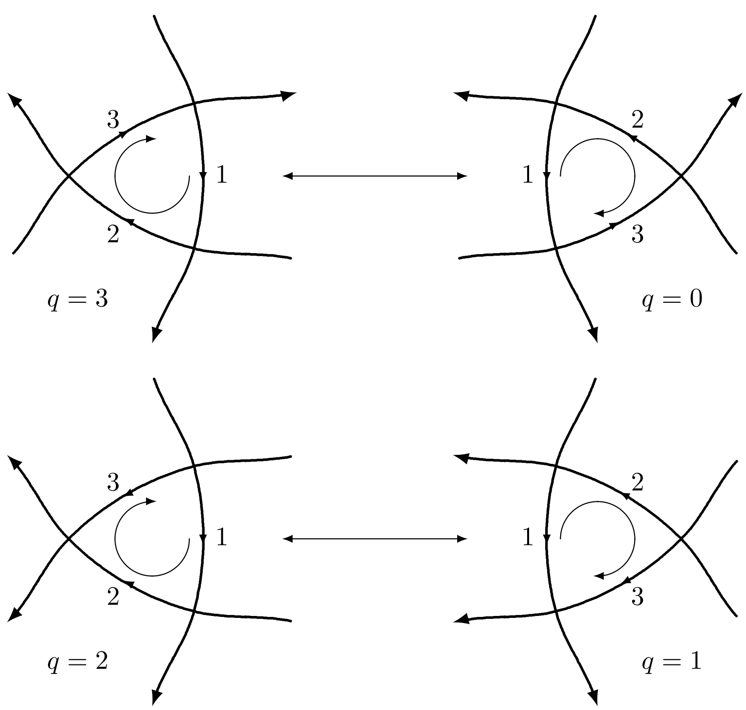


Figure 5. The vanishing triangle signs.

1.3.A. Theorem (Arnold [2]). *There exist three integers* $\mathrm{St}(C)$, $J^+(C)$, *and* $J^-(C)$ *corresponding to an arbitrary generic curve* C *which are uniquely defined by the following properties.*

(i) St, J^+ *and* J^- *are invariant under a regular homotopy in the class of generic curves.*

(ii) St *does not change under self-tangency perestroikas and* increase *by* 1 *under a positive triple point perestroika.*

(iii) J^+ *does not change under triple point and reverse self-tangency perestroikas and* increase *by* 2 *under positive direct self-tangency perestroika.*

(iv) J^- *does not change under triple point and direct self-tangency perestroikas and* decrease *by* 2 *under positive reverse self-tangency perestroika.*

(v) *On the standard curves* $K_0, K_1, K_2, \dots,$ *shown in Figure* 2, St, J^+ *and* J^- *take the following values*:

$$\begin{array}{lll} \mathrm{St}(K_0) = 0, & \mathrm{St}(K_{i+1}) = i & (i = 0, 1, 2, \dots); \\ J^+(K_0) = 0, & J^+(K_{i+1}) = -2i & (i = 0, 1, 2, \dots); \\ J^-(K_0) = -1, & J^-(K_{i+1}) = -3i & (i = 0, 1, 2, \dots). \end{array}$$

Remark. The normalization of St and $J^{\pm}$ which is fixed by the last property makes them to be additive with respect to the connected summation of curves.

1.4. Additional definitions. Let us consider a generic curve C. Its image gives us a partition of the plane $\mathbb{R}^2$ into the connected components of the image complement, the pieces of the curve between double points, and the double points. This is a stratification of the plane $\mathbb{R}^2$. The set of all k-dimensional strata we denote by Σ_k. As usual, all 0-strata are called *vertices*, 1-strata are called *edges*, and 2-strata are called *regions*. It is obvious that all regions are homeomorphic to the open disk except the one (which is called *exterior region*) which is homeomorphic to the open annulus.

Let us fix an initial point f on the curve C which differs from all self-intersection points, and a direction of move along the curve C at this point (that is an orientation). Let us enumerate all edges by numbers from 1 to $2n$ (where n is the number of vertices of the curve C) following the given direction and assigning 1 to the edge with the point f.

Let us consider an arbitrary vertex v. There are two edges which go into the vertex. Let them have numbers i and j such that the tangent vector to the edge i and the tangent vector to the edge j give us a positive orientation of the plane (see Figure 6). Let us assign the number $\mathrm{sign}(i - j)$ to the vertex v and to the edge j and the number $(-\mathrm{sign}(i - j))$ to the edge i. Now we assign the number $-\dfrac{\mathrm{sign}(i-j)}{2}$ to the region lying between the edges i and j and to the region which is opposite to that one, and the number $\dfrac{\mathrm{sign}(i-j)}{2}$ to the two other regions (see Figure 6).

Now for each region we add together all numbers which we get for this region from all vertices lying on the boundary. Therefore we assign some number to each stratum σ.

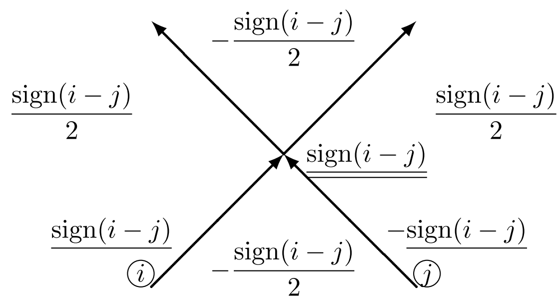


FIGURE 6. Weights of vertex, edges, and regions. The numbers in the circles are edge numbers. Twice underlined number corresponds to the vertex, single underlined numbers correspond to the edges, and not underlined at all to the regions.

1.4.A. DEFINITION. This number is called *weight* of the stratum σ and is denoted by $w(\sigma)$.

Let us define now for any stratum its *index* with respect to the oriented curve C.

1.4.B. DEFINITION. The *index* of a region $\sigma \in \Sigma_2$ is the total rotation number of the radius vector which connects an arbitrary interior point of the region σ to a point moving along the curve C. It is obvious that this number does not depend on the choice of the point in σ.

1.4.C. DEFINITION. The *index* of an edge $\sigma \in \Sigma_1$ is a half-sum of the indices of the two regions adjacent to σ.

1.4.D. DEFINITION. The *index* of a vertex $\sigma \in \Sigma_1$ is a quarter-sum of the indices of the four regions adjacent to σ.

We denote the index of an arbitrary stratum σ (with respect to the generic curve C) by $\operatorname{ind}_C(\sigma)$.

1.5. Formulas for St**.** Now we can formulate the main result of this paper. Let δ be the index of the edge which the initial point f belongs to. Then the following three formulas hold true:

$$\operatorname{St}(C) = \sum_{\sigma\in\Sigma_0} w(\sigma)\ \operatorname{ind}_C(\sigma) + \delta^2 - \frac{1}{4}, \tag{$*$}$$

$$\operatorname{St}(C) = \frac{1}{2}\sum_{\sigma\in\Sigma_1} w(\sigma)\ \operatorname{ind}_C^2(\sigma) + \delta^2 - \frac{1}{4}, \tag{$**$}$$

$$\operatorname{St}(C) = \frac{1}{3}\sum_{\sigma\in\Sigma_2} w(\sigma)\ \operatorname{ind}_C^3(\sigma) + \delta^2 - \frac{1}{4}. \tag{$***$}$$

Remark. It is easy to see that edge indices and region weights are half-integers. Nevertheless the strangeness is always an integer.

But before we prove the formulas $(*)$, $(**)$, and $(***)$ (see Section 2), let us reformulate them in other terms.

1.6. Ascending knot diagrams. It is possible to get another definition of vertex and region weights using an ascending diagram of a (trivial) knot with the curve C as a projection instead of an edge enumeration.

Indeed, consider an ascending diagram of a knot K drawn from the initial point f in the chosen direction. Then it follows from the definition of an ascending diagram that in a vertex v an edge with number i lies under the intersected edge with number j if and only if $i < j$ that is $\operatorname{sign}(i-j) < 0$. The definition 1.4.A of the vertex index implies that $w(v) = s(v)$ where $s(v)$ is the sign of the vertex v. It is show in Figure 7. Hence we get the following version of $(*)$:

$$\operatorname{St}(C) = \sum_{\sigma\in\Sigma_0} s(\sigma)\ \operatorname{ind}_C(\sigma) + \delta^2 - \frac{1}{4}. \tag{$*'$}$$

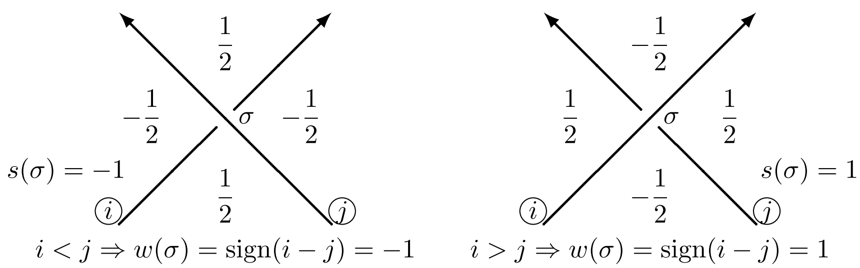


Figure 7. Vertex sign and weight in the case of ascending diagram. Vertex contribution into the region weights.

In order to rewrite $(***)$ we need to use Turaev's theory of shadows [5]. Since shadow theory is not used in this paper anywhere else, all following remarks are written here only for those who are interested in this theory.

The shadow of the knot K has the curve C as a projection. Let us denote the gleam of a region σ by $\operatorname{gl}(\sigma)$ (there are several gleam definitions in literature, so here we mean the latest one [6]). Since the contribution from each vertex into weight and gleam of a region σ are the same (see Figure 7 and [6, Chapter IX, Figure 3.4]), $\operatorname{gl}(\sigma) = w(\sigma)$. Now we can present $(***)$ in the following form:

$$\operatorname{St}(C) = \frac{1}{3}\sum_{\sigma\in\Sigma_2} \operatorname{gl}(\sigma)\ \operatorname{ind}^3_C(\sigma) + \delta^2 - \frac{1}{4} \tag{$***'$}$$

1.7. Gauss word of generic curves. Consider a generic curve C with a chosen initial point f. Let us assign some symbol to each its vertex. Let them be, say, a_1, $a_2, \dots, a_n$, where n is the number of vertices. Let us move along the oriented curve C from the point f, writing symbols which correspond to the passing vertices. We assign an exponent $(+1)$ to the symbol if the intersected curve branch is oriented from left to right with respect to the direction of our motion, and an exponent (-1) if it is oriented from right to left (see Figure 8 and [3]).

1.7.A. Definition. The obtained sequence is called *Gauss word* of the curve C and is denoted by $W(C)$.

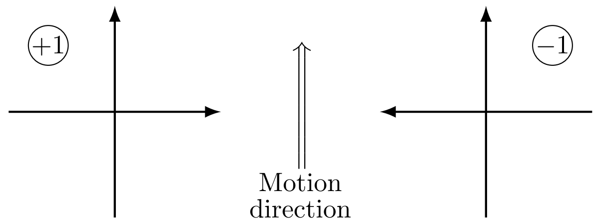
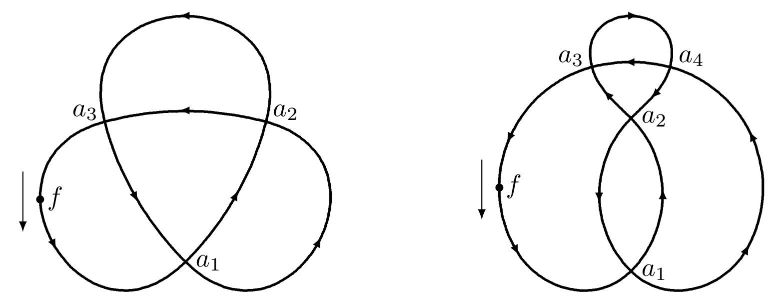


FIGURE 8. The vertex exponents in a Gauss word.

1.7.B. It is obvious that each symbol appears in $W(C)$ exactly two times but with the exponents with the opposite signs.

The simplest examples of the Gauss word are shown in Figure 9.



$W(C) = a_1^{+1} a_2^{-1} a_3^{+1} a_1^{-1} a_2^{+1} a_3^{-1}$ $\quad$ $W(C) = a_1^{+1} a_2^{-1} a_3^{-1} a_4^{+1} a_2^{+1} a_1^{-1} a_4^{-1} a_3^{+1}$

FIGURE 9. Examples of Gauss words.

Each edge except the first one, which we can consider separately, corresponds to a pair of symbols following one after another in such a word.

Let us understand now how to reconstruct from the Gauss word indices of edges and vertices. Let us choose the initial point f on some exterior edge (that is an edge which bounds the exterior region). Then the index of the first edge is $\pm\frac{1}{2}$, depending on the orientation of the curve C (see Figure 10).

FIGURE 10. The index of the first edge depending on the direction of motion. The numbers 0 and ± 1 are region indices and $\pm\frac{1}{2}$ are edge indices.

1.7.C. It is easy to see that the index of an edge after a vertex differs from the index of an edge before it exactly on the exponent of the vertex in the Gauss word (see Figure 11).

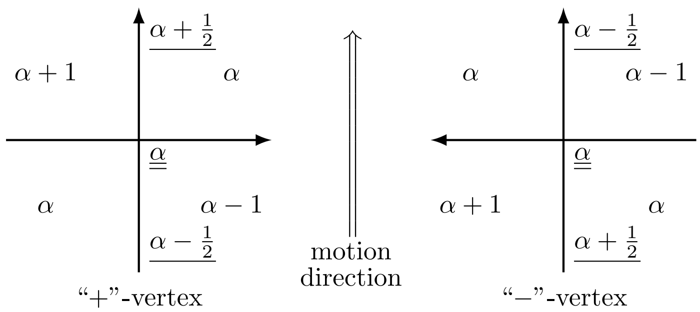


FIGURE 11. Changing of the edge index when passing through a vertex. Twice underlined numbers are vertex indices, single underlined ones are edge indices, and not underlined at all are region indices.

1.7.D. It follows from the same Figure 11 that the vertex index is equal to the index of the previous (in the Gauss word) edge to which we should either add (if in this place the vertex has exponent 1) or subtract (if it has exponent (-1)) number $\frac{1}{2}$. This ends the calculation of all vertex and edge indices using the Gauss word of the curve C.

We only need to describe now the weights of vertices and edges in the terms of the Gauss word of the curve C. Let us compare the definition 1.4.A of the weight of an arbitrary vertex v and the definition of the vertex exponent in the Gauss word. It is obvious that the weight $w(v)$ is equal to the exponent of the *first* appearance of the vertex v in the Gauss word.

In order to calculate the weight of an edge e, let us denote the vertex, which goes the edge to, by v. Let $\varepsilon(v)$ be the exponent which v has in the Gauss word right after the edge e. Then $w(e) = -\varepsilon(v)\, w(v)$. This formula allows us to calculate $w(v)$, since we already now how to find $w(v)$ (see above).

The complete proof of these facts amounts to the accurate consideration in Figure 6 cases whether i is greater or less than j and is left here to the reader.

Therefore $(*)$ and $(**)$ admits easy reformulation in the terms of the Gauss word $W(C)$ of the curve C.

Remark. It is possible to make similar reformulations in terms of the Gauss diagram of a curve C (see the definition in [2]), where we only need to mark in some way the vertex exponents. For instance, it can be done by orienting of chords on such a diagram from (-1) to $(+1)$.

2. Proof of the main formulas

2.1. Equivalence of the formulations. Let us show, first of all, that all three formulas are correct (or not correct) at the same time.

2.1.A. Lemma.

$$\sum_{\sigma_0\in\Sigma_0} w(\sigma_0)\ \mathrm{ind}_C(\sigma_0) = \frac{1}{2}\sum_{\sigma_1\in\Sigma_1} w(\sigma_1)\ \mathrm{ind}_C^2(\sigma_1) = \frac{1}{3}\sum_{\sigma_2\in\Sigma_2} w(\sigma_2)\ \mathrm{ind}_C^3(\sigma_2).$$

Proof. Since all three sums depend on the stratum weights linearly, it is enough to prove that contributions from each vertex v into these sums are the same. Indeed, let us look once again at Figure 6 (see Figure 12, where ε is, for simplicity, $\mathrm{sign}(i-j)$). Let α be index of the region lying between the edges with numbers i and j. Then indices of all other regions, edges and the vertex v are distributed as it is shown in Figure 12.

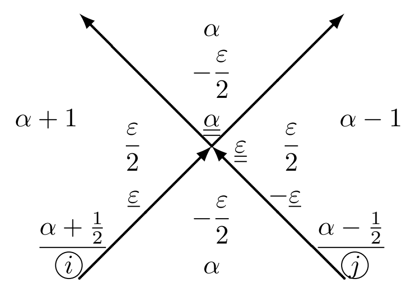


Figure 12. Indices and weights of vertices, edges, and regions. Numbers in the circles are edge numbers. Twice underlined number correspond to the index and weight of the vertex, single underlined ones to the edges, and not underlined at all to the regions.

Let us now calculate the contribution of the vertex v into each sum:

$$\begin{array}{ll} \text{1-st sum:} & \alpha\varepsilon, \\ \text{2-nd sum:} & \frac{1}{2}\left(\varepsilon\left(\alpha+\frac{1}{2}\right)^2-\varepsilon\left(\alpha-\frac{1}{2}\right)^2\right)=\frac{1}{2}(2\alpha\varepsilon)=\alpha\varepsilon, \\ \text{3-rd sum:} & \frac{1}{6}\left(\varepsilon(\alpha+1)^3+\varepsilon(\alpha-1)^3-2\varepsilon\alpha^3\right)=\frac{1}{6}(6\alpha\varepsilon)=\alpha\varepsilon. \quad \square \end{array}$$

Now we only need to prove $(*)$. We will do it in the ascending diagram terms (in 1.6 we already have proved the equivalence of the formulations), so we will actually prove $(*')$.

2.2. Independence from the initial point. Let us consider an ascending diagram $\mathcal{D}$ drawn starting from the initial point f. We denote the set of its vertices by V. Let

$$\Sigma_f(C) = \sum_{v\in V} s(v)\ \mathrm{ind}_C(v) + \delta^2 - \frac{1}{4},$$

where δ is, as above, the index of the edge which the point f belongs to. We need to prove that $\Sigma_f(C) = \mathrm{St}(C)$ independently of the choice of the initial point f on the curve C.

2.2.A. Lemma. *Let f_1 and f_2 be two points on the curve C. Then $\Sigma_{f_1}(C) = \Sigma_{f_2}(C)$.*

Proof. Let us denote by e_1 and e_2 the edges which f_1 and f_2 belong to. Let $s_1(v)$ and $s_2(v)$ be the signs of a vertex v depending on the chosen initial point. Let $\delta_1 = \mathrm{ind}_C(e_1)$ and $\delta_2 = \mathrm{ind}_C(e_2)$ (it is obvious that vertex, edge, and region indices do not depend on the choice of the initial point). Let $\Delta = \Sigma_{f_2}(C) - \Sigma_{f_1}(C)$.

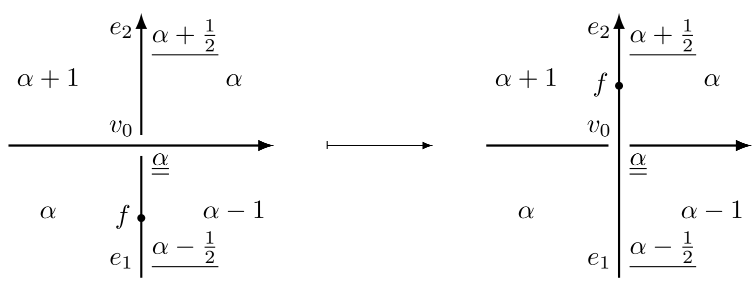


Figure 13. Changing of the initial point in the first case of orientation. Twice underlined numbers correspond to vertex indices, single underlined ones to edges, and not underlined at all to regions.

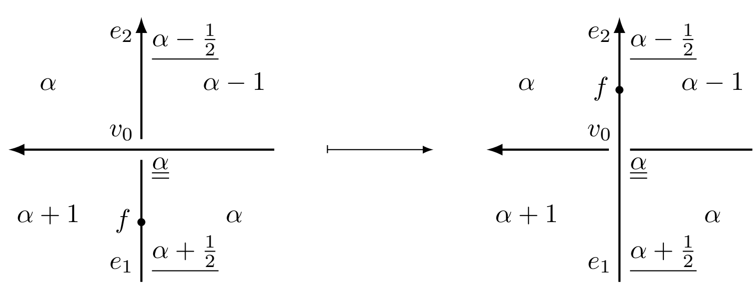


Figure 14. Changing of the initial point in the second case of orientation. Twice underlined numbers correspond to vertex indices, single underlined ones to edges, and not underlined at all to regions.

If the edges e_1 and e_2 coincide then we have nothing to prove. But if these edges are different, then it is enough to consider a case when one of them follows another. We can assume that e_2 follows e_1 and there is a vertex v_0 between them. It is easy to see that by change of the initial point, the diagram $\mathcal{D}$ changes only at the vertex v_0. Hence for any vertex $v \in V$ such that $v \neq v_0$ we have $s_1(v) = s_2(v)$. Therefore $\Delta = s_2(v_0)\operatorname{ind}_C(v_0) + \delta_2^2 - s_1(v_0)\operatorname{ind}_C(v_0) - \delta_1^2$.

It is shown in Figures 13 and 14 that there are two cases of index distribution which depend on the orientation of the intersected branch of the curve C.

In the first case (see Figure 13) $s_1(v_0) = 1$, $s_2(v_0) = -1$, $\operatorname{ind}_C(v_0) = \alpha$, $\delta_1 = \alpha - \frac{1}{2}$, and $\delta_2 = \alpha + \frac{1}{2}$. Therefore

$$\Delta = -\alpha + \left(\alpha + \frac{1}{2}\right)^2 - \alpha - \left(\alpha - \frac{1}{2}\right)^2 = 0.$$

The second case can be examined in a similar way. $\square$

Now we can change the notation from $\Sigma_f(C)$ to $\Sigma(C)$.

Remark. It follows from the proved Lemmas 2.1.A and 2.2.A that $(*)$, $(**)$, and $(***)$ can be considered as the definition of $\operatorname{St}(C)$.

2.3. Reidemeister moves. Since every knot represented by an ascending diagram isotopic to the trivial knot, Reidemeister Theorem [4] implies that the diagram $\mathcal{D}$ can be transformed into the standard diagram of the trivial knot by a finite sequence of the Reidemeister moves Ω_1, Ω_2 and Ω_3 (see Figure 15) and their inverses. It is quite obvious that during this transformation we can remain in the class of ascending diagrams.

There are also two kinds of moves Ω_1 and Ω_3 which are shown in Figure 16 and which we denote by Ω_1' and Ω_3'.

It follows from Figures 17 and 18 that Ω_1' and Ω_3' can be represented as a sequence of $\Omega_1^{\pm 1}$, $\Omega_2^{\pm 1}$, and $\Omega_3^{\pm 1}$.

2.3.A. Corollary. *It is possible to transform the diagram $\mathcal{D}$ into the standard diagram of the trivial knot by a finite sequence of moves Ω_1', Ω_2, and Ω_3 and their inverses. And, moreover, we can remain in the class of ascending diagrams (see Figure* 17*).*

2.3.B. Corollary. *It is possible to transform the diagram $\mathcal{D}$ into the standard diagram of the trivial knot by a finite sequence of moves Ω_1, Ω_2, and Ω_3' and their inverse. And, moreover, we can remain in the class of ascending diagrams (see Figure* 18*).*

Let us choose the initial point f on an arbitrary exterior edge. Lemma 2.2.A implies that the value of $\Sigma(C)$ does not depend on the choice of f, so we do not lose any generality. We say that a region *is involved in a Reidemeister move* if it lies close to the place of move and has either one edge in the case of Ω_1 and Ω_1' or two edges in the case of Ω_2 or three edges in the case of Ω_3 and Ω_3'.

2.3.C. Lemma. *It is possible to transform the diagram $\mathcal{D}$ into the standard diagram of the trivial knot in such a way that all regions involved into the Reidemeister moves do not have the point f on the boundary.*

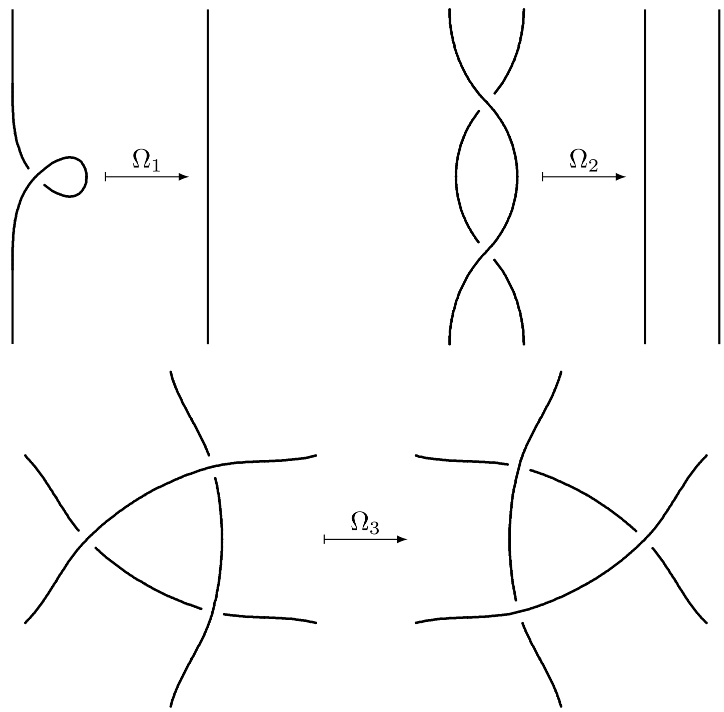


Figure 15. The three Reidemeister moves.

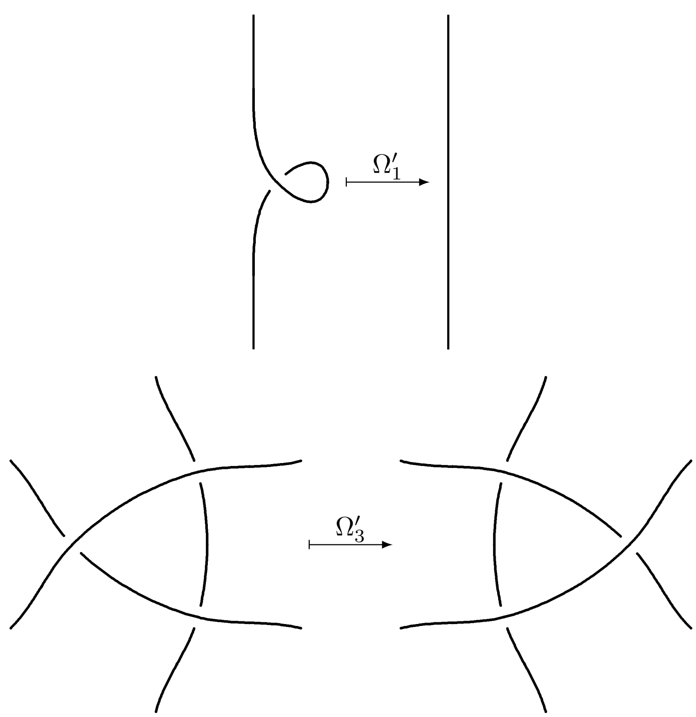


Figure 16. Additional Reidemeister moves.

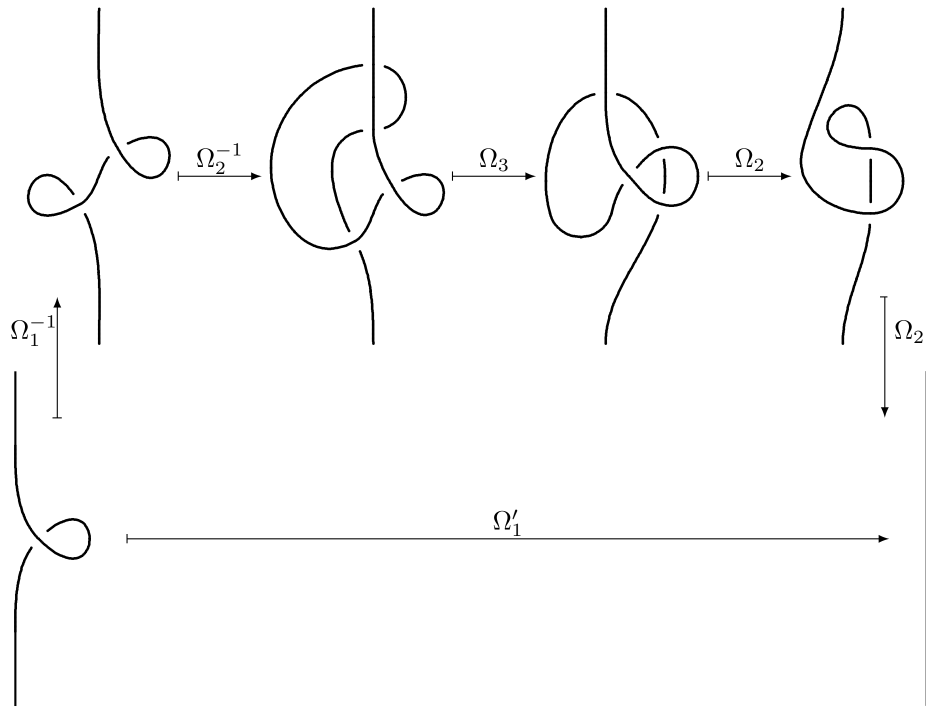


FIGURE 17. Expression of move Ω_1' through Ω_1^{-1}, $\Omega_2^{\pm 1}$ and Ω_3.

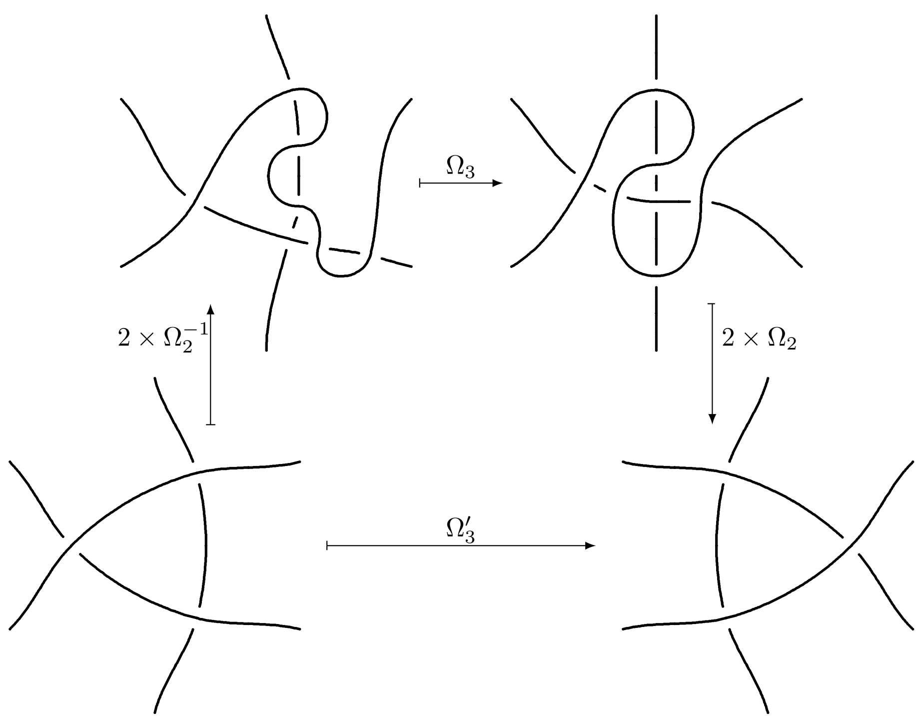


FIGURE 18. Expression of move Ω_3' through $\Omega_2^{\pm 1}$ and Ω_3.

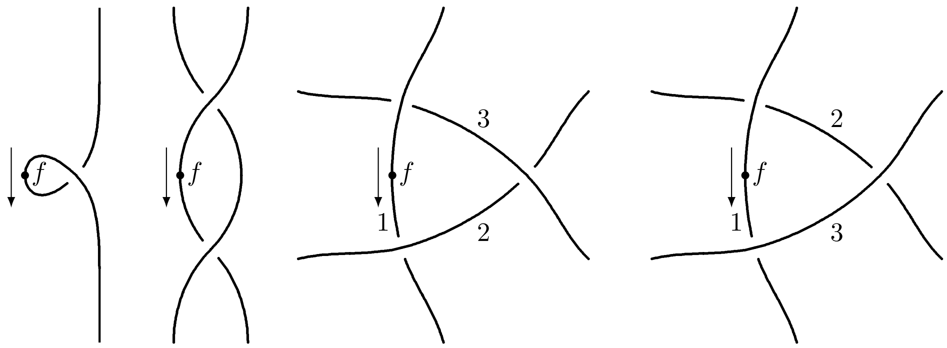


Figure 19. Regions with the point f on the boundary in the first case of orientation. The numbers show an order of moving along the edges.

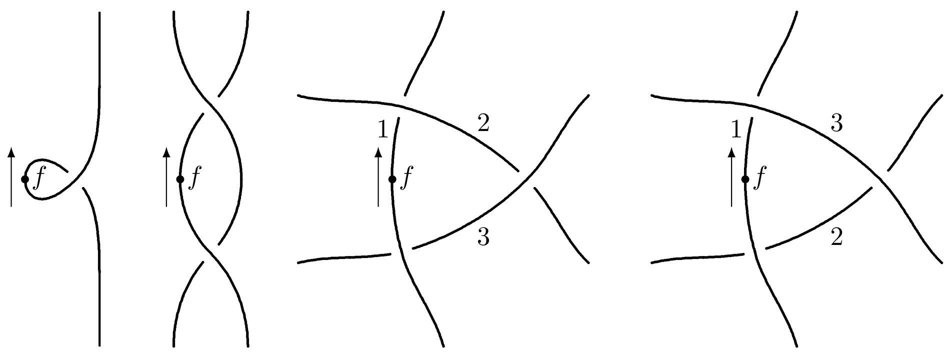


Figure 20. Regions with the point f on the boundary in the second case of orientation. The numbers show an order of moving along the edges.

Proof. Indeed, since f belongs to an exterior edge, all regions with one, two, or three edges and the point f on the boundary can look only as it is shown in Figure 19 or in Figure 20 (it depends on the direction of motion). In the first case they can be involved only in Ω_1' and Ω_3, and in the second one only in Ω_1 and Ω_3'. All we need now is to use either Corollary 2.3.B (in the first case) or Corollary 2.3.A (in the second one). $\square$

2.4. The end of the proof. Since for the standard diagram of the trivial knot the values of St and Σ are obviously the same, we only need to prove the following fact.

2.4.A. Theorem. *Under the Reidemeister moves Ω_1, Ω_1', Ω_2, Ω_3, Ω_3' and their inverses which have no involved regions with the point f on the boundary, the values of Σ and* St *change in the same way.*

Proof. Let us look at Figures 21, 22, 23, 24, and 25. What is shown are all possible orientations of diagram parts involved in moves Ω_1, Ω_1', Ω_2, Ω_3, and Ω_3', which do not contradict the ascendancy of the diagram. Numbers are indices of

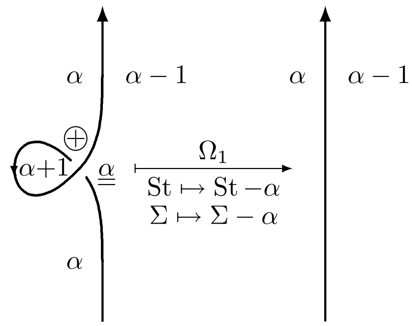


FIGURE 21. Behavior of St and Σ under move Ω_1.

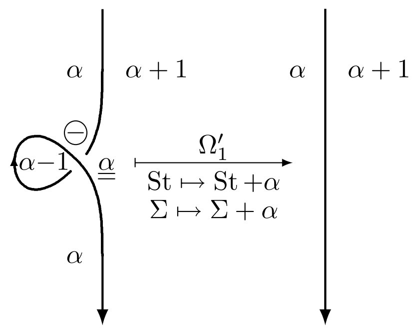


FIGURE 22. Behavior of St and Σ under move Ω_1'.

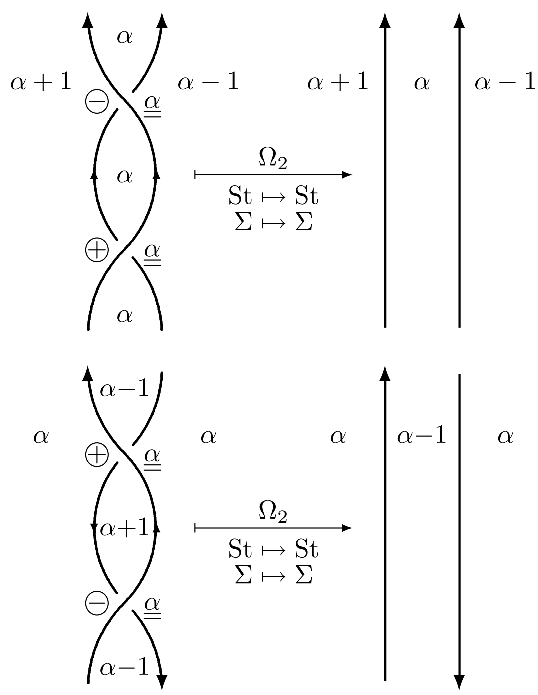


FIGURE 23. Behavior of St and Σ under move Ω_2.

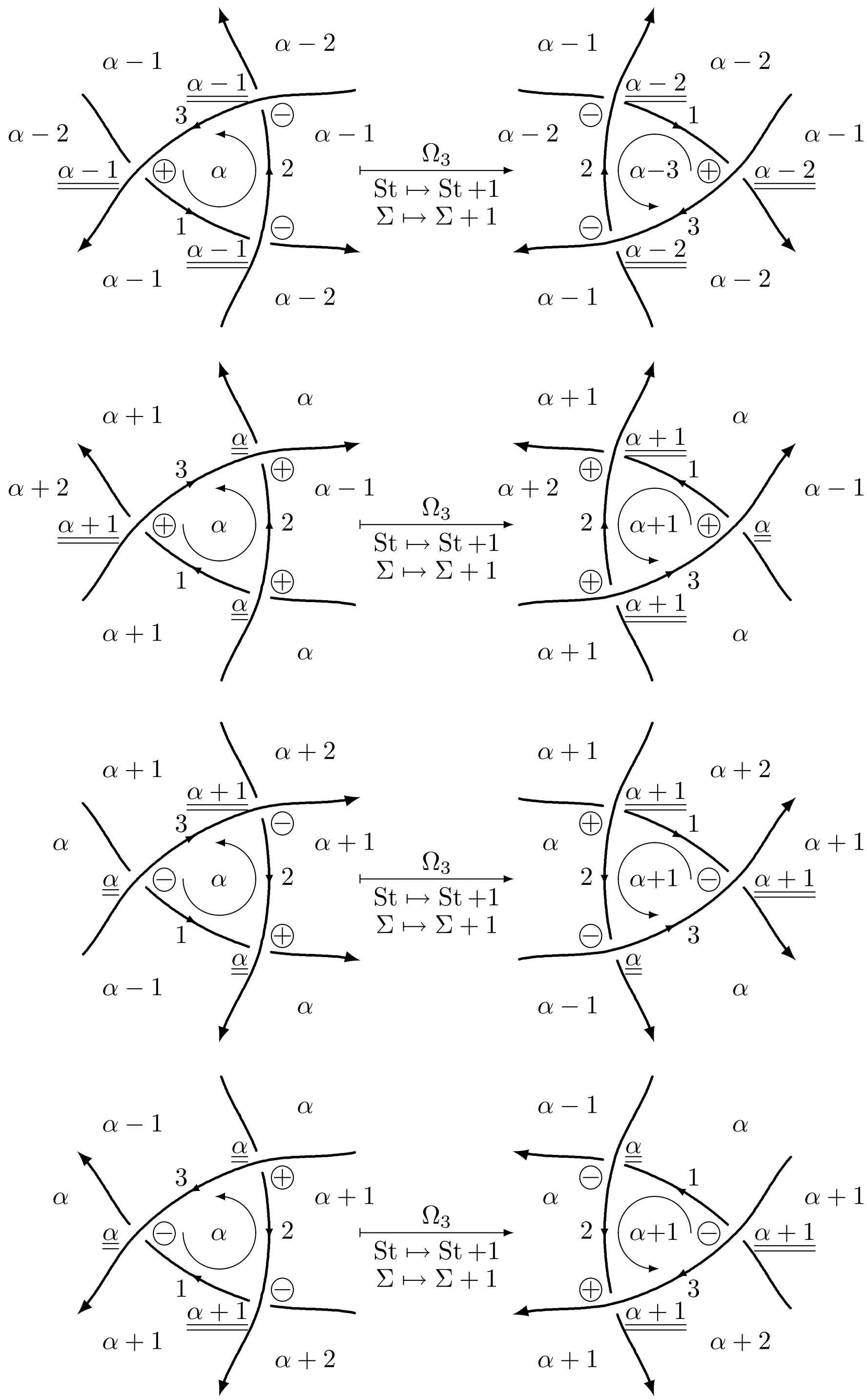


FIGURE 24. Behavior of St and Σ under move Ω_3.

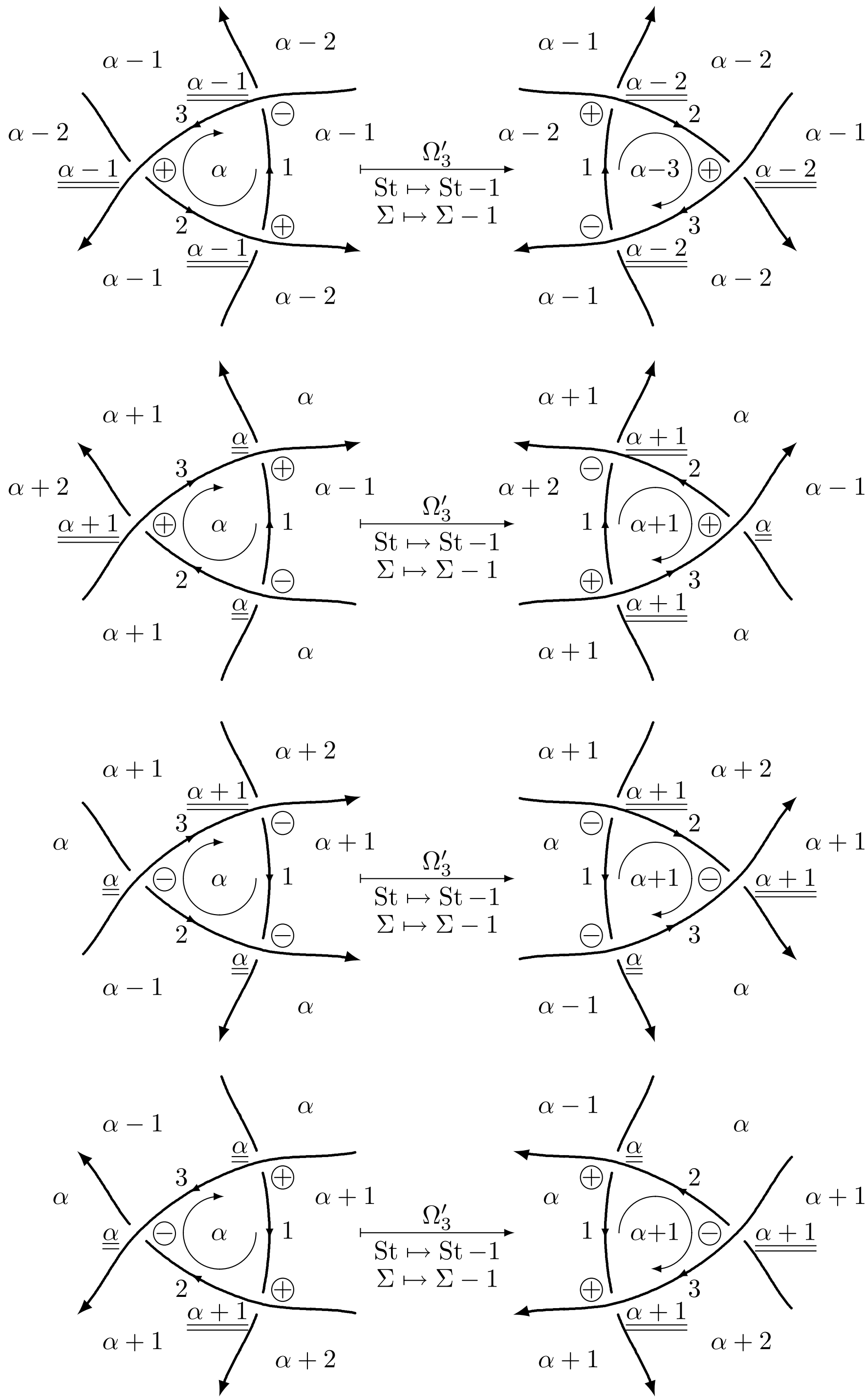


FIGURE 25. Behavior of St and Σ under move Ω_3'.

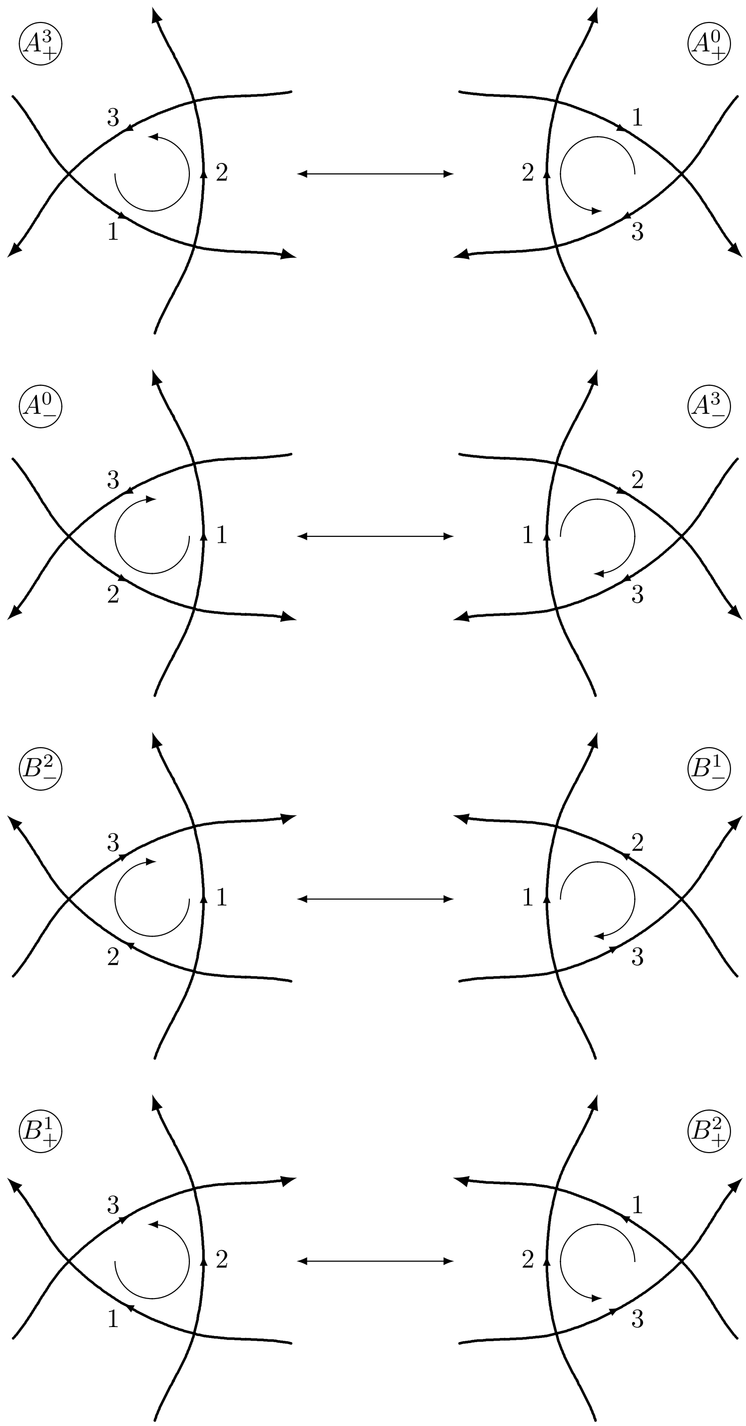


FIGURE 26. The four types of a triple point perestroika.

regions and vertices (underlined ones). Vertex signs are shown in circles close to the vertices. In the case of triangles, a cyclic edge order follows from the mutual position of under-crossings and over-crossing and is shown in Figures 24 and 25 by numbers 1, 2, 3 and arrows.

It is easy to check that St and Σ are changing in these cases exactly as is shown in the figures (the fact that St is changing by $\pm\alpha$ under moves Ω_1 and Ω_1', follows immediately from the "pushing away" formula proved in [2]). Hence St and σ are changing in the same way.

This ends the proof of the theorem as well as the proof of $(*)$, $(**)$ and $(***)$. $\square$

2.5. Additional remarks. It was shown by Arnold [2] that there are four types of a triple point perestroika which differ from each other by an orientation of the vanishing triangle edges and by a cyclic order given on this edges. These four perestroika types are respectively denoted by $A^3_+ \leftrightarrow A^0_+$, $A^0_- \leftrightarrow A^3_-$, $B^2_- \leftrightarrow B^1_-$, and $B^1_+ \leftrightarrow B^2_+$ (see Figure 26 and [2, Figure 17.]). It is easy to see after a comparison of Figures 24 and 25 with Figure 26 that the perestroikas $A^3_+ \leftrightarrow A^0_+$ and $B^1_+ \leftrightarrow B^2_+$ correspond to the Reidemeister move Ω_3 on the ascending diagram, and $A^0_- \leftrightarrow A^3_-$ and $B^2_- \leftrightarrow B^1_-$ correspond to Ω_3'.

We already have shown that as far as we go from one diagram to another, we can use either only Ω_3 move or Ω_3' move. In the case of curves we can similarly conclude that there is a generic regular homotopy between two generic curves such that all triple point perestroikas are either only $A^3_+ \leftrightarrow A^0_+$ and $B^1_+ \leftrightarrow B^2_+$ types or only $A^0_- \leftrightarrow A^3_-$ and $B^2_- \leftrightarrow B^1_-$ types.

3. Proof of the Arnold's conjectures

Starting at this point, we assume that the initial point f has been chosen on an exterior edge. It can be seen from Figure 10 that in this case $\delta = \pm\frac{1}{2}$.

3.1. Formula for the Whitney index. Let $u = 2\delta$.

3.1.A. Lemma. *For any generic curve C*

$$\operatorname{ind}(C) = u + \sum_{v\in V} w(v).$$

Proof. Let us observe that in the terms of the ascending diagram $\mathcal{D}$, the desired equality can be rewritten as $\operatorname{ind}(C) = u + \sum_{v\in V} s(v)$. We will prove it is this form.

It immediately follows from the definition of the sign $s(v)$ that $\sum_{v\in V} s(v) = \operatorname{writhe}(\mathcal{D})$, where $\operatorname{writhe}(\mathcal{D})$ is the self-linking number of a knot K with the diagram $\mathcal{D}$ and vertical framing. The curve C can be transformed into the standard curve with the same index (see Figure 2) by a sequence of perestroikas which correspond to the Reidemeister moves Ω_2, Ω_3, and Ω_3' and to their inverses on the Diagram $\mathcal{D}$.

Since neither $\operatorname{ind}(C)$ change under these perestroikas, nor $\operatorname{writhe}(\mathcal{D})$ under such moves, and u depends only on the direction of motion along the curve C (see Figure 10), we only need to check the formula $\operatorname{ind}(C) = u + \operatorname{writhe}(\mathcal{D})$ for the standard curves. But this is obvious (see Figure 27). $\square$

3.1.B. Corollary. *Since $u = \pm 1$ and for any vertex v we have $w(v) = \pm 1$, we get $|\operatorname{ind}(C)| \leqslant n + 1$ and $|\operatorname{ind}(C)| \equiv n + 1 \pmod 2$, where n is the number of vertices of the curve C.*

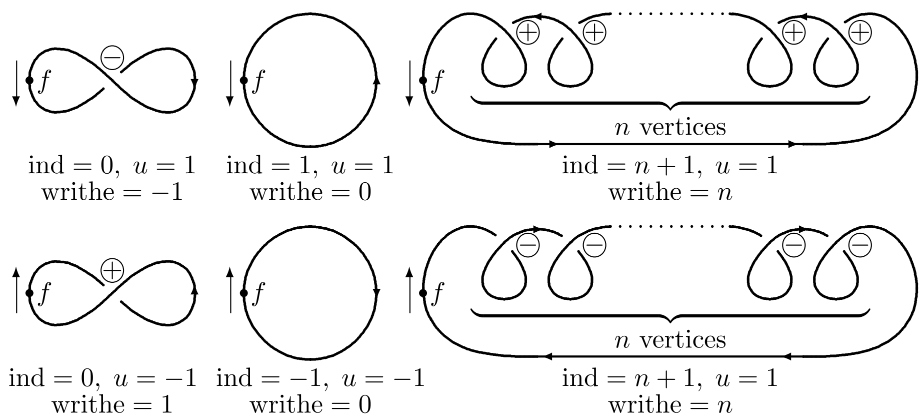


FIGURE 27. Checking the formula $\operatorname{ind}(C) = u + \operatorname{writhe}(\mathcal{D})$ for the standard curves.

Remark. Both Lemma 3.1.A Corollary 3.1.B were proved by Arnold [2] in slightly different terms.

3.2. Formulation of the Arnold's conjectures. Let n be the number of vertices of a generic curve C. It follows from Corollary 3.1.B that $|\operatorname{ind}(C)| = n + 1 - 2k$, where k is a nonnegative integer. This number is called *index defect* of the curve C or simply *defect*.

Let us denote by $\mathrm{St}_{\min}(n,k)$ and $\mathrm{St}_{\max}(n,k)$ the minimal and maximal possible values of St for curves with n vertices and index $n+1-2k$. In [2] Arnold has found formulas for $\mathrm{St}_{\min}$ and $\mathrm{St}_{\max}$ for $k = 0, 1, 2$. Moreover, he formulated the following conjectures:

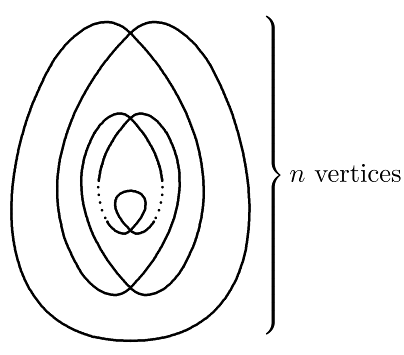


FIGURE 28. The curve A_{n+1}.

3.2.A. CONJECTURE (ARNOLD [2]). *When n is constant,* $\mathrm{St}_{\max}(n,k)$ *is monotonically decreasing as far as k is increasing.*

3.2.B. Conjecture (Arnold [2]). *When n is constant, the maximal possible value of* St *is attained only on the curve A_{n+1} (see Figure 28). Therefore* $\mathrm{St}(C) \leqslant \mathrm{St}(A_{n+1}) = \frac{n(n+1)}{2}$.

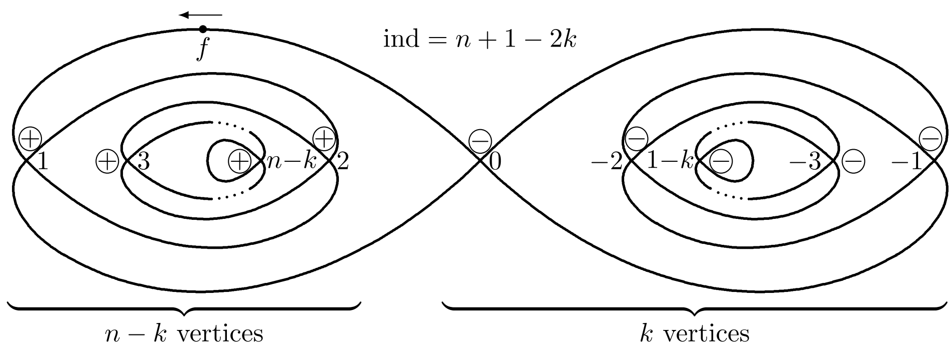
In this section we not only prove the Conjectures 3.2.A and 3.2.B but also find formulas for $\mathrm{St}_{\min}$ and $\mathrm{St}_{\max}$.

3.3. Formulas for $\mathrm{St}_{\min}$ and $\mathrm{St}_{\max}$. Let us consider a generic curve C with n vertices and index defect k.

3.3.A. Theorem.

$$\mathrm{St}(C) \leqslant \mathrm{St}_{\max}(n,k) = \frac{(n-k)(n-k+1)+(k-1)k}{2}.$$

Moreover this bound is sharp and can be attained only on the curves shown in Figure 29.



$$\mathrm{St} = \mathrm{St}_{\max}(n,k) = \sum_{i=1}^{n-k} i + \sum_{i=0}^{k-1} -(-i) = \frac{(n-k)(n-k+1)+(k-1)k}{2}$$

Figure 29. The curve maximizing the value of St.

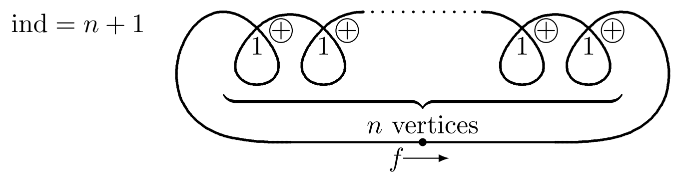
3.3.B. Theorem.

$$\mathrm{St}(C) \geqslant \mathrm{St}_{\min}(n,k) = \begin{cases} n & k=0 \\ -\frac{(k-1)(2n-3k+2)}{2} & k>0 \end{cases}.$$

Moreover this bound is sharp and can be attained only on the curves shown in Figure 30 (for $k=0$), in Figure 31 (for $k>0$), and in Figure 32 (for $n=2$ and $k=1$).

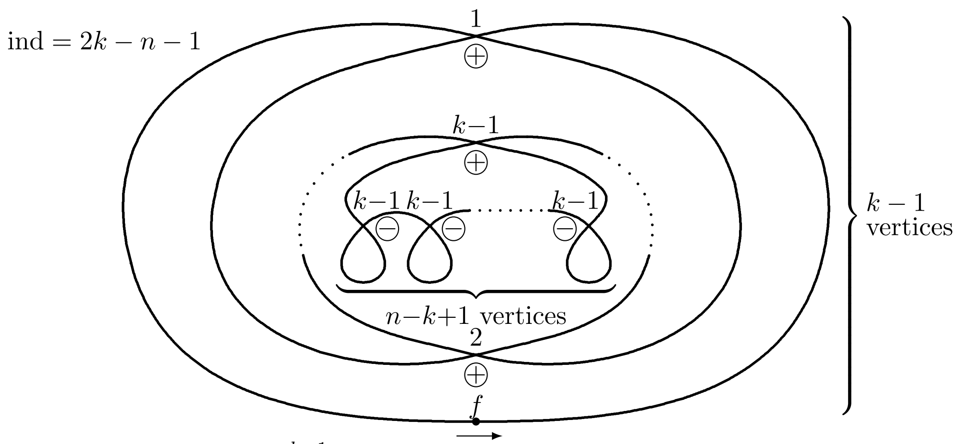
Remark. Similar results were obtained by F. Aicardi [1] in the case of tree-like curves (that are curves with the planar Gauss diagram).

3.3.C. It is obvious that the Conjecture 3.2.A is a direct corollary of Theorem 3.3.A.



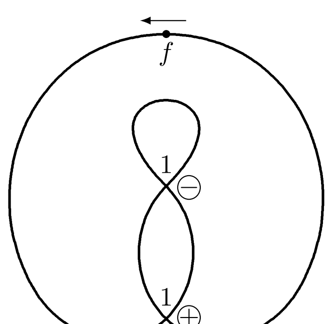
$$\mathrm{St} = \mathrm{St}_{\min}(n, 0) = n$$

FIGURE 30. The curves minimizing the value of St for $k = 0$.



$$\mathrm{St} = \mathrm{St}_{\min}(n, k) = \sum_{i=1}^{k-1} i - (k-1)(n-k+1) = -\frac{(k-1)(2n-3k+2)}{2}$$

FIGURE 31. The curves minimizing the value of St for $k > 0$.



$$\mathrm{St} = \mathrm{St}_{\min}(2, 1) = 1 - 1 = 0$$

FIGURE 32. Additional curve minimizing value of St for $n = 2$ and $k = 1$.

3.3.D. It can be seen from Theorem 3.3.A that for a constant n, the maximal possible value of St is $\frac{n(n+1)}{2}$ and it can be attained only for $k = 0$. In this case the curve shown in Figure 29 convert into the curve A_{n+1} (see Figure 28). Therefore the Conjecture 3.2.B also follows from Theorem 3.3.A.

3.3.E. It is easy to obtain from the Theorem 3.3.B that for a constant n the smallest value of $\mathrm{St}_{\min}(n, k)$ can be attained only for k close to $\frac{1}{3}n$.

We prove Theorems 3.3.A and 3.3.B slightly farther, after some auxiliary facts.

3.4. Additional facts.

3.4.A. LEMMA. *Let vertices v_1 and v_2 be the endpoints of the same edge. Then*

$$|\operatorname{ind}_C(v_1) - \operatorname{ind}_C(v_2)| \leqslant 1.$$

PROOF. See Figure 33, where all possible edge orientations are shown. $\square$

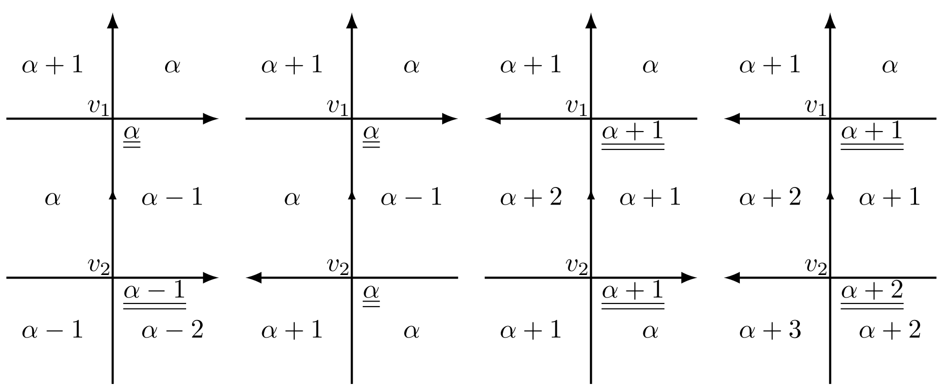


FIGURE 33. Indices of successive vertices.

3.4.B. COROLLARY. *Let $v_1, v_2 \in V$, $l_1 = \operatorname{ind}_C(v_1)$ $l_2 = \operatorname{ind}_C(v_2)$, and $l_1 \leqslant l_2$. Then for any l such that $l_1 \leqslant l \leqslant l_2$ there exists a vertex v with index l.*

PROOF. Let us move along the curve oriented C from v_1 to v_2. It follows from Lemma 3.4.A that during this motion vertex indices must take all intermediate values. $\square$

3.4.C. LEMMA. *If $u = 1$ then the index of the first vertex crossed is either 0 or 1, but if $u = -1$ then it is either (-1) or 0.*

PROOF. See Figure 34, where all possible edge orientations are shown. $\square$

Let us introduce the following notations: let n_+ be the number of vertices with positive weight, n_- with negative one, N_+ with negative both weight and sign, and N_- with negative ones. It is obvious that $N_+ \leqslant n_+$, $N_- \leqslant n_-$, and $n_+ + n_- = n$. Let us also denote by $i_{\max}$ and $i_{\min}$ the minimal and the maximal values of the vertex indices respectively.

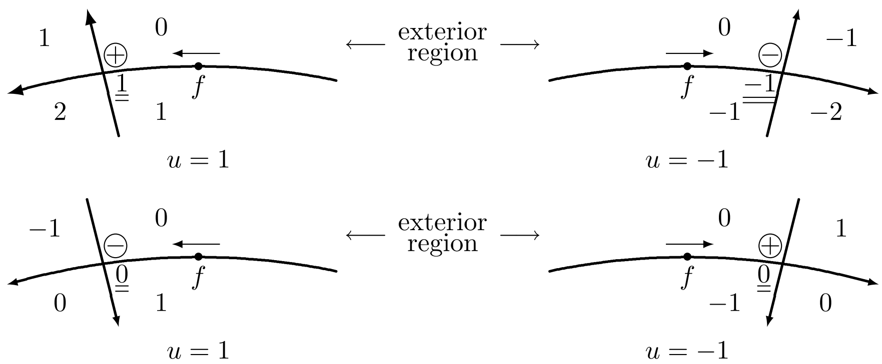


FIGURE 34. Index of the first vertex crossed.

3.4.D. LEMMA. *Let $v \in V$ and $l = \operatorname{ind}_C(v) \geqslant \max(0, u)$. Then there exists a vertex v_1 such that $\operatorname{ind}_C(v_1) = l$ and $w(v_1) = 1$.*

PROOF. Observe first of all that Lemma 3.4.C implies that the index of the first vertex crossed is not greater than $\max(0, u)$.

Consider the first moment as we move along the curve to be when a vertex index gets the value l. Denote this vertex by v_1. We want to prove that $w(v_1) = 1$. Indeed, if v_1 is the first vertex then $l = \operatorname{ind}_C(v_1) = \max(0, u)$, and for any value of u such a case corresponds to a vertex with positive weight (see Figure 34). But if v_1 is not the first one, then from Corollary 3.4.B we know that the index of the previous vertex is $l-1$. It is clear from Figure 33 that we can have such a situation only in the one case of edge orientation. Since we arrived at the vertex v_1 at the first moment, the weight $w(v_1)$ is exactly 1 (see Figure 33). □

3.4.E. COROLLARY. $i_{\max} \leqslant N_+$.

PROOF. Assume the contrary. Let there be a vertex v such that $\operatorname{ind}_C(v) > N_+$. Then it follows from Lemma 3.4.D that not all index values from 1 to $\operatorname{ind}_C(v)$ are realized, which contradicts Corollary 3.4.B. □

3.4.F. LEMMA. *Let $v \in V$ and $l = \operatorname{ind}_C(v) \leqslant \min(0, u)$. Then there exists a vertex v_1 such that $\operatorname{ind}_C(v_1) = l$ and $w(v_1) = -1$.*

PROOF. Similar to the proof of Lemma 3.4.D. □

3.4.G. COROLLARY. $i_{\min} \geqslant -N_-$.

PROOF. Similar to the proof of Corollary 3.4.E. □

Let us introduce some more notations. Let

$$\Sigma_+^+ = \sum_{\substack{w(v)=1 \\ \operatorname{ind}_C(v)>0}} w(v)\operatorname{ind}_C(v), \qquad \Sigma_+^- = \sum_{\substack{w(v)=1 \\ \operatorname{ind}_C(v)<0}} w(v)\operatorname{ind}_C(v),$$

$$\Sigma_-^+ = \sum_{\substack{w(v)=-1 \\ \operatorname{ind}_C(v)>0}} w(v)\operatorname{ind}_C(v), \qquad \Sigma_-^- = \sum_{\substack{w(v)=-1 \\ \operatorname{ind}_C(v)<0}} w(v)\operatorname{ind}_C(v),$$

where v belongs to the set of all vertices.

It is obvious that $\Sigma_+^+ \geqslant 0$, $\Sigma_+^- \leqslant 0$, $\Sigma_-^+ \leqslant 0$, and $\Sigma_-^- \geqslant 0$. Since $\delta = \pm\frac{1}{2}$, $(*)$ implies that $\mathrm{St}(C) = \Sigma_+^+ + \Sigma_+^- + \Sigma_-^+ + \Sigma_-^-$ (it is easy to see that a summand with zero index does not affect the sum in $(*)$).

3.5. Proof of Theorem 3.3.A.

3.5.A. Lemma. $\mathrm{St}(C) \leqslant \dfrac{N_+(N_+ + 1) + N_-(N_- + 1)}{2}$.

Proof. Since $\Sigma_+^- \leqslant 0$ and $\Sigma_-^+ \leqslant 0$, we get $\mathrm{St}(C) \leqslant \Sigma_+^+ + \Sigma_-^-$. It follows from Corollary 3.4.B and Lemma 3.4.C that vertex indices must take all values from 1 to $i_{\max}$. Since Corollary 3.4.E implies that $i_{\max} \leqslant N_+$, we get

$$\begin{aligned}\Sigma_+^+ &\leqslant \sum_{i=1}^{i_{\max}} i + i_{\max}(N_+ - i_{\max}) = \frac{i_{\max}(i_{\max}+1) + 2i_{\max}(N_+ - i_{\max})}{2} \\ &= \frac{i_{\max} + 2N_+ i_{\max} - (i_{\max})^2}{2} = \frac{i_{\max} - (N_+ - i_{\max})^2 + N_+^2}{2} \\ &\leqslant \frac{N_+ + N_+^2}{2} = \frac{N_+(N_+ + 1)}{2}.\end{aligned}$$

Similarly,

$$\Sigma_-^- \leqslant \frac{N_-(N_- + 1)}{2}.$$

Therefore,

$$\mathrm{St}(C) \leqslant \frac{N_+(N_+ + 1) + N_-(N_- + 1)}{2}. \quad \square$$

The end of the proof of Theorem 3.3.A. Assume that $u = 1$ (it does not affect the computation of St). From Lemma 3.4.C and Corollary 3.4.B we get that if there is a vertex with a negative index then there is one with a zero index. Hence it follows from Lemma 3.4.F that there exists a vertex with a negative weight and zero index. Therefore $N_- \leqslant \max(0, n_- - 1)$. Since $N_+ \leqslant n_+$, from Lemma 3.5.A we see that

$$\begin{aligned}\mathrm{St}(C) &\leqslant \frac{n_+(n_+ + 1) + \max(0, n_- - 1)(\max(0, n_- - 1) + 1)}{2} \\ &= \frac{n_+(n_+ + 1) + (n_- - 1)n_-}{2}.\end{aligned}$$

However $n + 1 - 2k = |\,\mathrm{ind}(C)| = |u + n_+ - n_-|$ and $n_+ + n_- = n$. Hence either $n_+ = n - k, n_- = k$, or $n_+ = k - 1, n_- = n - k + 1$. In the both cases we get that

$$\mathrm{St}(C) \leqslant \frac{(n-k)(n-k+1) + k(k-1)}{2}. \tag{+}$$

The sharpness of the bound obtained easily follows from Figure 29, where all necessary calculations are done. So now we only need to prove that the maximum cannot be attained on other curves.

Let us denote the curve shown in Figure 29 by $A_{n-k,k}$. It can be seen from the proof of Lemma 3.5.A and from the calculations above that equality in (+) can be achieved only for $i_{\max} = N_+ = n_+$ and $-i_{\min} = N_- = \max(0, n_- - 1)$. It is possible only in the case when vertex indices take all values from $(-n_- + 1)$ to n_+ exactly once. Let us prove by induction on $N = N_+ + N_-$ that in this case $C \cong A_{n_+,n_-}$.

The initial statement for $N = 0$ is obvious since then we have $n_+ = 0$ and either $n_- = 0$ (hence $C \cong K_1$) or $n_- = 1$ (hence $C \cong K_0$). Let us prove now our statement for arbitrary $N \geqslant 1$ using an inductive hypothesis that is already proved for $N - 1$.

Since $N \geqslant 1$ we have either $n_+ \geqslant 1$ or $n_- \geqslant 2$. Consider first the case $n_+ \geqslant 1$. Let us denote by v the vertex with the index n_+. Remark, that all vertex indices are not greater than n_+ and there is only one vertex with the index $(n_+ - 1)$. Therefore Lemma 3.4.A implies that as we move along the curve C we have to cross the vertex v two times successively. Hence C forms a small curl in the vertex v (see Figure 35).

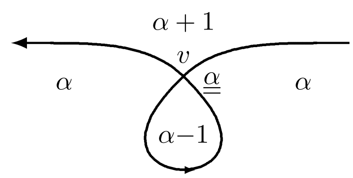


Figure 35. The vertex with the maximal index.

Let us remove the curl and denote the resulting curve by C_1. It is obvious that C_1 has $(n-1)$ vertices whose indices take all values from $(-n_- + 1)$ to $(n_+ - 1)$. Inductive hypothesis implies that $C_1 \cong A_{n_+-1,n_-}$. Therefore C can be obtained from A_{n_+-1,n_-} by adding a vertex and a small curl in it. Since $w(v) = 1$, the curl is oriented as shown in Figure 35. In this case the index of v is equal to the index of the region where the curl has been made (see Figure 35). Hence the index of the region is $\mathrm{ind}_C(v) = n_+$. There is a unique region for the curve A_{n_+-1,n_-} with such a property, therefore there is unique place where we can add a vertex with a curl. It is easy to see that the obtained curve is A_{n_+,n_-} (see Figure 29).

The case $n_- \geqslant 2$ can be examined similarly.

In order to finish the proof of Theorem 3.3.A we only need to remark that $A_{n-k,k}$ can be obtained from $A_{k-1,n-k+1}$ by reflection with respect to the vertical axis. Hence these curves are actually the same. $\square$

3.6. Proof of Theorem 3.3.B. Similarly to the proof of Theorem 3.3.A we assume that $u = 1$. Let

$$m(\alpha,\beta) = \begin{cases} \frac{(\beta-1)\beta}{2} - \alpha(\beta-1) & \alpha \geqslant \beta \\ \frac{\alpha(\alpha+1)}{2} - \alpha\beta & \alpha < \beta \end{cases}.$$

3.6.A. Lemma. *Let α and β be integers such that $\alpha \geqslant 0$ and $\beta \geqslant 1$. Then*

$$\min_{\substack{0\leqslant a\leqslant\alpha \\ 0\leqslant b\leqslant\beta-1}} \left(\frac{a(a+1)}{2} + \frac{b(b+1)}{2} - a(\beta-1-b) - b(\alpha-a) \right) \geqslant m(\alpha,\beta),$$

and equality can be attained only in the following three cases:

1) $\alpha \geqslant \beta,\ a = 0,\ b = \beta - 1$; 2) $\alpha = 0,\ b = 0$; 3) $\alpha \geqslant \beta,\ a = 0,\ \beta = 1$.

3.6.B. Lemma. *Let α and β be integers such that $\alpha \geqslant 1$ and $\beta \geqslant 0$. Then*

$$\min_{1 \leqslant a \leqslant \alpha} \left(\frac{a(a-1)}{2} + \alpha - a\beta \right) \geqslant m(\alpha, \beta),$$

and equality can be attained only in the following two cases:

$$1)\ \beta = 0,\ a = 1; \qquad 2)\ \alpha \leqslant \beta,\ a = \alpha.$$

We prove both these lemmas slightly farther in the Appendix.

In order to finish the proof let us examine two cases: one where there is a vertex with zero index and the other without.

The first case. Suppose that there is a vertex with index 0. Lemma 3.4.F implies $n_- \geqslant 1$ and $N_- \leqslant n_- - 1$. It follows from Lemma 3.4.C and Corollary 3.4.B that vertex indices must take all values from 1 to $i_{\max}$. Hence

$$\Sigma_+^+ \geqslant \sum_{i=1}^{i_{\max}} i + 1(N_+ - i_{\max}) \geqslant \frac{i_{\max}(i_{\max}+1)}{2}.$$

Similarly

$$\Sigma_-^- \geqslant \frac{-i_{\min}(-i_{\min}+1)}{2}.$$

It is also easy to see that

$$\Sigma_+^- \geqslant i_{\min}(n_+ - N_+) \geqslant i_{\min}(n_+ - i_{\max})$$

and

$$\Sigma_-^+ \geqslant -i_{\max}(n_- - 1 - N_-) \geqslant -i_{\max}(n_- - 1 + i_{\min}).$$

Therefore

$$\mathrm{St}(C) \geqslant \frac{i_{\max}(i_{\max}+1)}{2} - \frac{i_{\min}(-i_{\min}+1)}{2} + i_{\min}(n_+ - i_{\max}) - i_{\max}(n_- - 1 + i_{\min}).$$

Corollaries 3.4.E and 3.4.G imply $0 \leqslant i_{\max} \leqslant N_+ \leqslant n_+$ and $0 \leqslant -i_{\min} \leqslant N_- \leqslant n_- - 1$. Hence it follows from Lemma 3.6.A that $\mathrm{St}(C) \geqslant m(n_+, n_-)$.

Since $n + 1 - 2k = |\operatorname{ind}(C)| = |u + n_+ - n_-|$ and $n_+ + n_- = n$, either $n_+ = n - k, n_- = k$ or $n_+ = k - 1, n_- = n - k + 1$. Remark also that since $n_+ \geqslant 0$ and $n_- \geqslant 1$, we get $k \geqslant 1$.

Let $S_1 = m(n-k, k)$, $S_2 = m(k-1, n-k+1)$. Since $n-k \geqslant k$ and $k-1 < n-k+1$,

$$S_1 = \frac{k(k-1)}{2} - (k-1)(n-k),$$
$$S_2 = \frac{k(k-1)}{2} - (k-1)(n-k+1).$$

Then $S_1 - S_2 = k - 1 \geqslant 0$ for all $k \geqslant 1$. Therefore

$$\mathrm{St}(C) \geqslant S_2 = \frac{(k-1)(-2n+3k-2)}{2}. \qquad (++)$$

We only need to find all possibilities when the equality in this formula holds.

Lemma 3.6.A implies that there are only three cases when it is so:

$$1)\ n_+ \leqslant n_-,\ i_{\max} = 0,\ -i_{\min} = n_- - 1; \qquad 2)\ n_+ = 0,\ i_{\min} = 0;$$
$$3)\ n_+ \leqslant n_- = 1,\ i_{\max} = 1.$$

Since $S_1 > S_2$ for all $k > 1$, equality in $(++)$ in the cases 1) and 3) is possible only for $k = 1$. But in the second case $n_+ = 0$. Therefore k also must be 1. Thus in all three cases we get that equality is possible only for $k = 1$ and $i_{\max} = i_{\min} = 0$. There is unique curve which has all vertex indices equal to 0 (see Figure 36). For this curve the cases 1) and 3) coincide and differ from the case 2) by a choice of the initial point f (see Figure 36). Remark that the curve shown in Figure 36 is the partial case of the curve in Figure 31 when $k = 1$. This completes the consideration of the first case. □

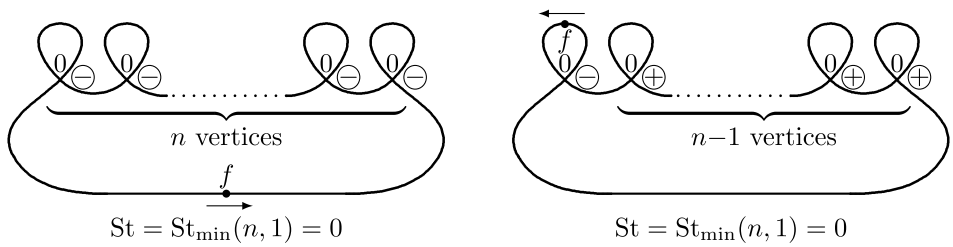


FIGURE 36. The curve with all vertex indices equal to 0.

THE SECOND CASE. Suppose now that there is no vertex with zero index. Then by Lemma 3.4.C and Corollary 3.4.B we get that all vertices must have strictly positive indices. Remark, that in this case $N_+ = n_+$, $N_- = 0$, and $i_{\min} = 1$. It is obvious that $\Sigma_+^- = \Sigma_-^- = 0$. As in the First case we can get that

$$\Sigma_+^+ \geqslant \sum_{i=1}^{i_{\max}} i + 1(n_+ - i_{\max}) = \frac{i_{\max}(i_{\max}-1)}{2} + n_+$$

and

$$\Sigma_-^+ \geqslant -i_{\max} n_-.$$

Then

$$\mathrm{St}(C) \geqslant \frac{i_{\max}(i_{\max}-1)}{2} + n_+ - i_{\max} n_-.$$

Corollary 3.4.E implies $1 \leqslant i_{\max} \leqslant N_+ = n_+$. Therefore from Lemma 3.6.B we get $\mathrm{St}(C) \geqslant m(n_+, n_-)$.

In a similar way (as in the First case) we have that if $k = 0$ then

$$\mathrm{St}(C) \geqslant m(n, 0) = n, \qquad (+++)$$

but if $k \geqslant 1$ then

$$\mathrm{St}(C) \geqslant \min(S_1, S_2) = -\frac{(k-1)(2n-3k+2)}{2}. \qquad (++')$$

Lemma 3.6.B implies that there are only two cases when equality can be attained in $(+++)$ and $(++')$:

$$1)\ n_- = 0,\ i_{\max} = 1; \qquad 2)\ n_- \geqslant n_+,\ i_{\max} = n_+.$$

If $k = 0$ only the first case is possible. Since all vertices now have index 1 and $n_- = 0$, it is easy to observe that there exists a unique curve satisfying these conditions (see Figure 30).

Now we only need to consider the case when $k \geqslant 1$. Since $n_- > 0$ it implies that equality in $(++')$ can be attained only for $n_+ \leqslant n_-$ and $i_{\max} = n_+$. If we get the minimum when $n_+ = k-1$ and $n_- = n-k+1$ then the indices of all vertices with positive weight must take all values from 1 to n_+ exactly once and the indices of all vertices with negative weight are equal to n_+. It easy to check that the curve has to look like the one shown in Figure 31 (the proof is the same as in the case of Theorem 3.3.A).

Let us examine now the last case when the minimum is attained for $n_+ = k$ and $n_- = n-k$. It is shown in the proof of the First case that this situation is possible only for $k = 1$. $n_+ \leqslant n_-$ implies that $n = 2$ and $n_+ = n_- = 1$. There exist only one curve satisfying these conditions (see Figure 32). $\square$

Theorem 3.3.B is thus completely proved. $\square$

Appendix

Proof of Lemma 3.6.A. Let

$$f(a,b) = \left(\frac{a(a+1)}{2} + \frac{b(b+1)}{2} - a(\beta-1-b) - b(\alpha-a)\right).$$

Since $\frac{\partial^2 f}{\partial a \partial a} = \frac{\partial^2 f}{\partial b \partial b} = 1$ and $\frac{\partial^2 f}{\partial a \partial b} = \frac{\partial^2 f}{\partial b \partial a} = 2$, the determinant of the Hessian of the function f is (-3). This number is negative, hence the function f never has local extremal points and it is enough for us to check the desired equality only for boundary values of a and b.

(1) $f(0,b) = \dfrac{b(b+1)}{2} - b\alpha$.
If $\alpha \geqslant \beta$ then since $b \leqslant \beta - 1$, we get $2\alpha - \beta - b > 0$. Therefore,

$$\begin{gathered}(\beta-1-b)(2\alpha-\beta-b) \geqslant 0\\ \Downarrow \\ b(b+1) - 2b\alpha \geqslant \beta(\beta+1) - 2\alpha(\beta-1)\\ \Downarrow \\ \frac{b(b+1)}{2} - b\alpha \geqslant \frac{\beta(\beta+1)}{2} - \alpha(\beta-1),\end{gathered}$$

and it is easy to see that we can have equality only for $b = \beta - 1$.

But if $\alpha < \beta$ then $\left(b-\alpha+\frac{1}{2}\right)^2 \geqslant \dfrac{1}{4}$ (since $(b-\alpha)$ is an integer). Then

$$\frac{\left(b-\alpha+\frac{1}{2}\right)^2}{2} - \frac{1}{8} \geqslant \alpha(\alpha-\beta)$$

$$\Downarrow$$

$$\frac{\left(b-\alpha+\frac{1}{2}\right)^2}{2} - \frac{1}{8} - \frac{\alpha(\alpha-1)}{2} \geqslant \frac{\alpha(\alpha+1)}{2} - \alpha\beta$$

$$\Downarrow$$

$$\frac{b(b+1)}{2} - b\alpha \geqslant \frac{\alpha(\alpha+1)}{2} - \alpha\beta,$$

and it is obvious that we can have equality only for $\alpha = b = 0$.

(2) Similarly we get that $f(a,0) = \dfrac{a(a+1)}{2} - a(\beta-1) \geqslant m(\alpha,\beta)$, but in this case equality can be attained only for $\alpha \geqslant \beta = 1$ and $a = 0$.

(3) $f(\alpha,b) = \dfrac{\alpha(\alpha+1)}{2} + \dfrac{b(b+1)}{2} - \alpha(\beta-1-b) > \dfrac{\alpha(\alpha+1)}{2} - \alpha(\beta-1-b)$ (since the case $b=0$ has already been considered, we can assume that $b \geqslant 1$ and, therefore, $\dfrac{b(b+1)}{2} > 0$).

If $\alpha \geqslant \beta$, then $\dfrac{\alpha(\alpha+1)}{2} - \alpha(\beta-1-b) \geqslant \dfrac{\beta(\beta-1)}{2} - \alpha(\beta-1)$ (since $\alpha \geqslant 0$ and $b \geqslant 0$).

But if $\alpha < \beta$, then $\dfrac{\alpha(\alpha+1)}{2} - \alpha(\beta-1-b) \geqslant \dfrac{\alpha(\alpha+1)}{2} - \alpha\beta$.

Therefore, $f(\alpha,b) > m(\alpha,\beta)$.

(4) Similarly, $f(a,\beta-1) = \dfrac{a(a+1)}{2} + \dfrac{\beta(\beta-1)}{2} - b(\alpha-a) > m(\alpha,\beta)$.

Lemma 3.6.A is thus completely proved. □

Proof of Lemma 3.6.B. Let us remember that we need to prove that if $\alpha \geqslant 1$ then for any $1 \leqslant a \leqslant \alpha$

$$\frac{a(a-1)}{2} + \alpha - a\beta \geqslant m(\alpha,\beta).$$

If $\alpha > \beta$ then $\left(a-\beta-\frac{1}{2}\right)^2 \geqslant \dfrac{1}{4}$ (since $(a-\beta)$ is integer). Then

$$\frac{\left(a-\beta-\frac{1}{2}\right)^2}{2} - \frac{1}{8} \geqslant \beta(\beta-\alpha)$$

$$\Downarrow$$

$$\frac{\left(a-\beta-\frac{1}{2}\right)^2}{2} - \frac{1}{8} - \frac{\beta(\beta+1)}{2} + \alpha \geqslant \frac{\beta(\beta-1)}{2} - \alpha(\beta-1)$$

$$\Downarrow$$

$$\frac{a(a+1)}{2} + \alpha - a\beta \geqslant \frac{\beta(\beta-1)}{2} - \alpha(\beta-1),$$

and it is obvious that we can have equality only for $\beta = 0$, $a = 1$.

But if $\alpha \leqslant \beta$ then $2\beta + 1 - a - \alpha > 0$. Therefore

$$(a-\alpha)(a+\alpha-1-2\beta) \geqslant 0$$
$$\Downarrow$$
$$2\beta(\alpha-a) + a(a-1) - \alpha(\alpha-1) \geqslant 0$$
$$\Downarrow$$
$$\frac{a(a+1)}{2} + \alpha - a\beta \geqslant \frac{\alpha(\alpha+1)}{2} - \alpha\beta,$$

and equality is possible only for $a = \alpha$.

This completes the proof of Lemma 3.6.B since for $\alpha = \beta$ we have $\frac{\beta(\beta-1)}{2} - \alpha(\beta-1) = \frac{\alpha(\alpha+1)}{2} - \alpha^2 = \frac{\alpha(\alpha+1)}{2} - \alpha\beta$. $\square$

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E-mail address: Shurik@math.uu.se, Shurik@ucrmath.ucr.edu