UNIVERSITY OF CALIFORNIA
RIVERSIDE

Vassiliev invariants of degree one of knots and links in $\mathbb{R}^1$- and S^1-fibrations

A Dissertation submitted in partial satisfaction
of the requirements for the degree of

Doctor of Philosophy
in
Mathematics
by
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June, 1997

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ACKNOWLEDGMENTS

I am deeply grateful to Oleg Viro for inspiration of this work and all the enlightening discussions. I am thankful to Francesca Aicardi, Tobias Ekholm, Thomas Fiedler and Michael Polyak for all the valuable discussions we had.

ABSTRACT OF THE DISSERTATION

Vassiliev invariants of degree one

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by

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Doctor of Philosophy, Graduate Program in Mathematics

University of California, Riverside, June, 1997

Professor Oleg Viro, Chairperson

As it is well-known, all Vassiliev invariants of degree one of a knot $K \subset \mathbb{R}^3$ are trivial. There are nontrivial Vassiliev invariants of degree one, when the ambient space is not $\mathbb{R}^3$. Recently, T. Fiedler introduced such invariants of a knot in an $\mathbb{R}^1$-fibration over a surface F. They take values in the free $\mathbb{Z}$-module generated by all the free homotopy classes of loops in F.

Here I generalize his invariants to the most refined Vassiliev invariant of degree one. I also construct a similar invariant of two-component links. It generalizes the linking number.

I also construct a similar first degree Vassiliev invariant of an oriented knot in an S^1-fibration and a Seifert fibration over a surface. It takes values in a quotient of the group ring of the first homology group of the total space of the fibration. It gives rise to an invariant of wave fronts on surfaces and orbifolds, related to the Bennequin type invariants of the Legendrian curves, studied by V. Arnold and M. Polyak. Formulas expressing these relations are written down.

I calculate Turaev's shadow for the Legendrian lifting of a wave front. This allows one to use all the invariants, known for shadows, in the case of wave fronts.

Contents

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Most of the proofs in this text are postponed till the last section.

Everywhere in this text $\mathbb{R}^1$- and S^1-fibrations mean locally-trivial fibrations with fibers, homeomorphic to $\mathbb{R}^1$ and S^1, respectively.

In this paper the multiplicative notation for the addition in the first homology group is used. The zero homology class is denoted by e. The reason for this is, that we have to deal with the integer group ring of the first homology group. For a group G, I denote the group of all formal half-integer linear combinations of elements of G by $\frac{1}{2}\mathbb{Z}[G]$.

We work in the differential category.

1. Introduction

In [5] M. Polyak suggested a quantization $l_q(L) \in \frac{1}{2}\mathbb{Z}[q, q^{-1}]$ of the Bennequin invariant of a generic cooriented oriented wave front $L \subset \mathbb{R}^2$. In this paper I construct an invariant $S(L)$, which is, in a sense, a generalization of $l_q(L)$ to the case of a wave front on an arbitrary surface F.

In the same paper [5] M. Polyak introduced V. Arnold's [3] J^+ type invariant of a front L on an oriented surface F. It takes values in $H_1(ST^*F, \frac{1}{2}\mathbb{Z})$. I show that $S(L) \in \frac{1}{2}\mathbb{Z}[H_1(ST^*F)]$ is a splitting of this invariant, in the sense, that it is taken to Polyak's invariant under the natural mapping $\frac{1}{2}\mathbb{Z}[H_1(ST^*F)] \to H_1(ST^*F, \frac{1}{2}\mathbb{Z})$.

Further I generalize $S(L)$ to the case, when L is a wave front on an orbifold.

Invariant $S(L)$ is constructed in two steps. The first one is lifting of L to the Legendrian knot λ in the S^1-fibration $\pi : ST^*F \to F$. The second step can be applied to any knot in an S^1-fibration, and it involves the structure of the fibration in a crucial way. This step allows one to define the S_K invariant of a knot K in the total space of an S^1-fibration. Since ordinary knots are considered up to a rougher equivalence relation (ordinary isotopy versus Legendrian isotopy), in order for S_K to be well-defined, it has to take values in a factor of $\mathbb{Z}[H_1(N)]$. This invariant is generalized to the case of a knot in a Seifert fibration, and this allows one to define $S(L)$ in the case of wave fronts on orbifolds.

All these invariants are Vassiliev invariants of degree one in an appropriate sense.

For each of these invariants I introduce its version, taking values in the group of formal linear combinations of the free homotopy classes of oriented curves in the total space of the corresponding fibration.

The first invariants of this kind were constructed by T. Fiedler [4] in the case of a knot K in a $\mathbb{R}^1$-fibration over a surface and by F. Aicardi in the case of a generic oriented cooriented wave front $L \subset \mathbb{R}^2$. In this text I generalize T. Fiedler's invariants to the most refined Vassiliev invariant of degree one of knots and links in $\mathbb{R}^1$- and S^1-fibrations and discuss the connection between all these invariants and S_K.

The space ST^*F is naturally fibered over a surface F with a fiber S^1. In [9] V. Turaev introduced a shadow description of a knot K in an oriented three dimensional manifold N, fibered over a surface with a fiber S^1. A shadow presentation of a knot K is a generic projection of K, enriched by an assignment of numbers to regions. It describes a knot type modulo a natural action of $H_1(F)$. It appeared to be a very useful tool. Many invariants of knots in S^1-fibrations, in particular quantum state sums, can be expressed as state sums for their shadows. In this work I construct shadows of Legendrian liftings of wave fronts. This allows one to use any invariant already known for shadows in the case of wave fronts.

However, in this paper shadows are used just for the purpose of depicting knots in S^1-fibrations.

2. Invariants of knots of knots and links

2.1. Basic definitions. We say, that a one-dimensional submanifold L of a total space N^3 of a fibration $p : N^3 \to M^2$ is *generic with respect to p* , if $p\big|_L$ is a generic immersion. An immersion of a one-manifold into a surface is said to be *generic*, if it has neither self-intersection points of multiplicity greater than two, nor self-tangency points, and at each double point its branches are transversal to each other. An immersion of (a circle) S^1 to a surface is called a *curve*.

Let F be a connected smooth two-dimensional surface (not necessarily compact or orientable) and $p : E \to F$ be an $\mathbb{R}^1$-fibration with oriented total space E. Let $K \subset E$ be a (smooth) oriented knot, in general position with respect to p.

2.1.A. Definition (Fiedler [4]). Let q be a double point of $p(K)$. Fix an orientation on the fiber $E_q = p^{-1}(q)$. This determines, which of the two branches of K, intersecting E_q, is over-crossing and which is under-crossing. Define local writhe $\omega(q)$ to be one if the three-frame (under-crossing, over-crossing, fiber E_q) agrees with the orientation on E and minus one, otherwise. (It is easy to check, that this definition does not depend on the choice of an orientation on E_q.)

2.2. Direct generalization of Fiedler's invariants. In [4] T. Fiedler introduced invariants of a knot K in an oriented total space of an $\mathbb{R}^1$-fibration $p : \mathbb{R} \to F$. As it follows from [8], these invariants can be expressed through a more symmetrical invariant U_K, introduced below. If F is oriented, then U_K also can be expressed through Fiedler's invariants. The formulas, expressing them through each other (see [8]), involve the values of all these invariants on some fixed knot homotopic to K.

Let $q \in p(K)$ be a crossing point. Split the curve $p(K)$ at q according to the orientation and obtain two oriented loops on F (see Figure 1).

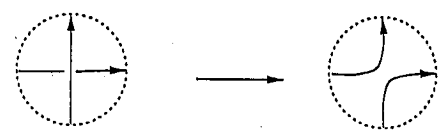

FIGURE 1. Splitting of $p(K)$.

2.2.A. DEFINITION. For a crossing point q of $p(K)$ denote by $\xi_1(q)$ and $\xi_2(q)$ the free homotopy classes of the two loops, created by splitting at q. Let H be the free $\mathbb{Z}$-module generated by the set of all the free homotopy classes of oriented loops on F. Define $U_K \in H$ by the following formula, where the summation is taken over all the crossings, such that none of the two loops, created by splitting, is homotopic to a trivial loop.

$$U_K = \sum_{\substack{q \in Q \\ \xi_1(q), \xi_2(q) \neq e}} \omega(q)\Big(\xi_1(q) + \xi_2(q)\Big) \tag{2.1}$$

2.2.B. THEOREM. *U_K is an isotopy invariant of the knot K.*

The proof is straightforward. One checks, that U_K does not change under all the oriented versions of the three Reidemeister moves.

2.2.C. Similarly to [4], one can introduce a version of U_K, which takes values in $\mathbb{Z}[H_1(F)]$. To obtain it, one substitutes $\xi_1(q)$ and $\xi_2(q)$ in (2.1) by the homology classes, realized by the corresponding loops. The summation should be made over the set of all the double points of $p(K)$, such that none of the two loops created by the splitting is homologous to 0.

2.2.D. Let $p : E \to F$ be an $\mathbb{R}^1$- or an S^1-fibration over a surface. Let $K \subset E$ be a knot generic with respect to p and q be a crossing point of $p(K)$. The modification of pushing of one branch of K through the other along a fiber E_q is called the *modification (of the knot) along the fiber E_q*.

2.2.E. THEOREM. *(Cf. Fiedler [4]) Let q be a crossing point of $p(K)$. Denote by i and j the free homotopy classes of the two loops, created by splitting of $p(K)$ at q according to the orientation. Under the modification along E_q the jump of U_K is*

$$\begin{cases} \pm 2\Big(i+j\Big), & \text{if } i,j \neq e, \\ 0, & \text{otherwise.} \end{cases} \tag{2.2}$$

Here the sign depends on $\omega(q)$.

The proof is straightforward.

2.2.F. COROLLARY. *U_K is a Vassiliev invariant of degree one.*

To get the proof, one notices, that the first derivative of U_K depends only on the free homotopy classes of the two loops, that appear, if one splits the singular knot (with one transverse double point) at the double point according to the orientation. Hence, the second derivative of U_K is identically zero.

2.3. The most refined Vassiliev invariant of degree one.

2.3.A. Unfortunately U_K appears to be not the most refined Vassiliev invariant of degree one of a knot in an $\mathbb{R}^1$-fibration. To show this, we construct two knots K_1 and K_2 and a first degree Vassiliev invariant $\tilde{U}_K$, such that $U_{K_1} = U_{K_2}$, and $\tilde{U}_{K_1} \neq \tilde{U}_{K_2}$.

2.3.B. DEFINITION OF $\tilde{U}_K$. Let Γ be an oriented figure eight graph (bouquet of two circles), V_Γ be its vertex and E^1_Γ and E^2_Γ be its edges. Set S to be a set of free homotopy classes of mappings of Γ into F, factorized by an orientation preserving involution of Γ. Let G be the free $\mathbb{Z}$-module generated by S. For a double point q of $p(K)$ put $G_q \in S$ to be the class of the mapping of Γ to F, which sends V_Γ to q, $E^1_\Gamma \cup E^2_\Gamma$ onto $p(K)$, according to the orientations of the edges, and is injective on the complement of the preimages of the double points of $p(K)$. Let $S' \subset S$ be those classes, for which none of the two loops of the figure eight graph is homotopic to a trivial loop. Define $\tilde{U}_K \in G$ by the following formula, where the summation is taken over the set of all the crossings q of $p(K)$, such that $G_q \in S'$.

$$\tilde{U}_K = \sum_{\{q \in p(K) | G_q \in S'\}} \omega(q) G_q \, . \tag{2.3}$$

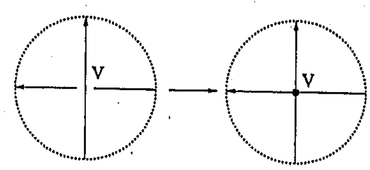

FIGURE 2. Figure eight graph corresponding to a double point.

Similarly to 2.2.F one checks, that $\tilde{U}_K$ is a Vassiliev invariant of degree one.

Let F be a disc with two holes. Let K_1 be the knot, shown on Figure 3, and K_2 be the knot obtained from K_1 by modifications along fibers over the crossing points u and v. (The two shaded discs on Figure 3 are the two holes.) One can easily check, that $U_{K_1} = U_{K_2}$, but $\tilde{U}_{K_1} \neq \tilde{U}_{K_2}$.

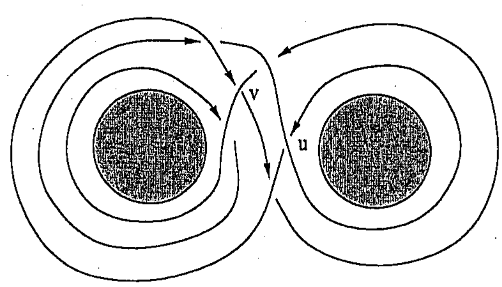

FIGURE 3. Two knots with different $\tilde{U}_K$ and equal U_K values.

The following theorem shows, that $\tilde{U}_K$ invariant is the most refined Vassiliev invariant of degree one.

2.3.C. THEOREM. *Let $v_1(K)$ be any Vassiliev invariant of degree one. It induces a mapping v_1^* : $G \to \mathbb{Z}$, which maps a class of the projection of a singular knot K' to $v_1(K')$. Fix some knot K_f. Then for any knot K, which is free homotopic to K_f*

$$v_1(K) = v_1(K_f) + \frac{1}{2} v_1^*(\tilde{U}_K - \tilde{U}_{K_f}) \tag{2.4}$$

2.3.D. *Proof of Theorem* 2.3.C.

One can obtain K from K_f by a sequence of isotopies and modifications along fibers. Both v_1 and $\tilde{U}_K$ are invariant under isotopy. If under a modification along a fiber $\tilde{U}_K$ jumps by $2G_q$, then v_1 jumps by $2v_1^*(G_q)$. (Clearly v_1 does not jump under modification along a fiber, for which one of the two loops of G_q is homotopic to a trivial loop.) The total jump of $\tilde{U}_K$ under the homotopy is

$\tilde{U}_{K_f} - \tilde{U}_K$. Thus the corresponding jump of v_1 invariant is $v_1(K_f) - v_1(K) = \frac{1}{2}v_1^*(\tilde{U}_{K_f} - \tilde{U}_K)$ and we proved the theorem. □

It is natural to take the simplest knot in the corresponding class as the K_f knot. Unfortunately, there is no canonical way to choose one.

As a corollary of Theorem 2.3.C we get, that for any Vassiliev invariant of degree one v_1 and two homotopic knots K_1 and K_2, equality $\tilde{U}_{K_1} = \tilde{U}_{K_2}$ implies $v_1(K_1) = v_1(K_2)$.

The following theorem, characterizes the range of values $\tilde{U}_K$.

2.3.E. THEOREM. *For a singular knot K_s (whose only singularity is a transverse double point) denote by $\bar{K}_s$ the free homotopy class of knots, that contains K_s. For a knot K denote by G_K the submodule of G generated by the classes of the projections of singular knots K_s, such that $K \in \bar{K}_s$.*

I: *Let K and K' be two oriented knots, representing the same free homotopy class. Then $\tilde{U}_K$ and $\tilde{U}_{K'}$ are congruent modulo the $2G_K$ submodule.*

II: *Let K be an oriented knot, $\tilde{U}$ be an element of G, such that it is congruent to $\tilde{U}_K$ modulo the $2G_K$ submodule. Then there exists an oriented knot K', such that:*

a) K and K' represent the same free homotopy class.

b) $\tilde{U}_{K'} = \tilde{U}$.

For the proof of Theorem 2.3.E see Section 8.1.

2.3.F. There is a natural mapping $\phi : G \to H$, which maps $g \in G$ to a formal sum of the free homotopy classes of the two loops of g. Clearly, $\phi(\tilde{U}_K) = U_K$. (The $\ker(\phi)$ is nontrivial and this is the reason, why U_K is not the most refined invariant of degree one.) Using ϕ and Theorem 2.3.E we obtain the following characterization of the range of values of U_K.

I: If K and K' are two oriented knots representing the same free homotopy class, then U_K and $U_{K'}$ are congruent modulo the $\phi(2G_K)$ submodule.

II: Let K be an oriented knot, U be an element of H, such that it is congruent to U_K modulo the $\phi(2G_K)$ submodule. Then, there exists an oriented knot K', such that:

a) K and K' represent the same free homotopy class.

b) $U_{K'} = U$.

2.3.G. The $\tilde{U}_K$ gives rise to a homotopy invariant of an oriented curve C on a surface F. To introduce it, we take an oriented $\mathbb{R}^1$-fibration E over F, and embed F with our curve C on it into E as a zero section. Slightly deforming our curve at the crossing points we obtain a knot K_C in

E. There is a natural $\mathbb{Z}$-module homomorphism π from G onto $\bar{G}$ the free $\mathbb{Z}_2$-module generated by the free homotopy classes from the set S' (see 2.3.B). From the first part of 2.3.E it is clear, that $\bar{U}_C = \pi(\tilde{U}_{K_C})$ is a homotopy invariant of C. For a free homotopy class of figure eight graph l the coefficient of it in $\bar{U}_C$ counts if the number of graphs (homotopic to l), one obtains by choosing some crossing point of C as a graph vertex is odd or even.

2.4. Partial linking polynomial. Let Θ be an annulus. Consider a solid torus T embedded into $\mathbb{R}^3$, and a projection $p : \mathbb{R}^3 \to \mathbb{R}^2$, such that $\mathrm{Im}(p|_T)$ is homeomorphic to Θ. Let $K \subset T$ be an oriented knot, in general position with respect to p. We denote by $i_1(q)$ and $i_2(q)$ the homology classes in $H_1(\Theta)$ of the two loops, that are created by splitting of $p(K)$ at the double point q. As $H_1(\Theta) = \mathbb{Z}$ we can consider $i_1(q)$ and $i_2(q)$ as integer numbers.

2.4.A. Definition (Aicardi [2]). Set partial linking polynomial $A(K)$ (originally in [2] it was denoted by $s[K]$) to be a finite Laurent polynomial, defined by the following formula

$$A(K) = \sum_{\substack{q \in Q \\ i_1(q), i_2(q) \neq 0}} \frac{1}{2}\Big(\omega(q)(t^{i_1(q)} + t^{i_2(q)})\Big) \tag{2.5}$$

Below by a_i we denote the coefficient of t^i in $A(K)$.

2.4.B. The set of all the free homotopy classes of oriented loops in Θ coincides with $H_1(\Theta)$. One can easily see, that U_K is mapped to $2A(K)$ under the natural isomorphism $\psi : H \to \mathbb{Z}[q, q^{-1}]$.

The fact, that $\pi_1(T) = \mathbb{Z}$ allows one to reconstruct an element $g \in G$ from the homology classes of the two loops of it. Thus, in this case $\tilde{U}_K$ invariant can also be reconstructed from $A(K)$.

2.4.C. *(Aicardi* [2]*).* Let $h \in \mathbb{Z}$ be the image of $[p(K)]$ (the homology class realized by $p(K)$) under the natural identification of $H_1(\Theta)$ with $\mathbb{Z}$. Then $a_0 = a_h = 0$ and $a_i = a_{h-i}$ for an arbitrary $i \in \mathbb{Z}$.

2.4.D. One can see, that the very definition of $A(K)$ depends on the embedding of T into $\mathbb{R}^3$. It is well known, that the group of orientation preserving autohomeomorphisms of T, factorized by isotopy relation, is isomorphic to $\mathbb{Z}$. It is generated by the class of an autohomeomorphism Φ, that extends a positive Dehn twist along a meridian of ∂T. That is cutting T along a meridional disc, twisting by 2π in a positive direction and gluing back. Replacement of the embedding of T to $\mathbb{R}^3$ by an isotopic one does not change $A(K)$. Embeddings of all isotopic classes can be obtained from the given one by a composition with Φ^n for some $n \in \mathbb{Z}$.

Let $A'(K)$ be the partial linking polynomial calculated, after we compose our embedding of T with Φ. Put

$$\Delta A(K) = A'(K) - A(K) \tag{2.6}$$

Let $h \in \mathbb{Z}$ be the homology class realized by $p(K)$.

2.4.E. THEOREM.

$$\Delta A(K) = \begin{cases} -|h|(t^1 + t^2 + \cdots + t^{h-1}), & \text{if } h > 0 \\ -|h|(t^{-1} + t^{-2} + \cdots + t^{h+1}), & \text{if } h < 0 \\ 0, & \text{if } h = 0 \end{cases} \tag{2.7}$$

For the proof of Theorem 2.4.E see Section 8.2.

As we can make the composition of our embedding with Φ^n, for any $n \in \mathbb{Z}$, we obtain the following.

2.4.F. $A(K)$ as an invariant of the topological pair $K \subset T$ is defined up to an addition of $\Delta A(K)$. Thus, an $A(K)$ invariant of a knot K, could be said to be in a canonical form, if it satisfies the following conditions:

$$\begin{cases} 0 \le a_1 < h & \text{for } h > 0, \\ 0 \le a_{-1} < |h| & \text{for } h < 0, \end{cases} \tag{2.8}$$

If $h = 0$, then $A(K)$ is always in the canonical form.

2.4.G. THEOREM. *Fix $h \in \mathbb{Z}$. Let P_h be a subset of all finite Laurent polynomials $\sum_{i=i_1}^{i_2} p_i t^i$, satisfying the following properties:*

a) $p_0 = p_h = 0$

b) $\forall j \in \mathbb{Z} \quad p_j = p_{h-j}$

c) if $h = 2k$ for some $k \in \mathbb{Z}$ then p_k is odd.

Then P_h is the range of values of the partial linking polynomial for knots homologous to h.

For the proof of Theorem 2.4.G see Section 8.3.

2.5. Invariant of links.

2.5.A. DEFINITION OF U_L. Let $p : E \to F$ be an $\mathbb{R}^1$-fibration, of an oriented space E over a surface. Let Γ be an oriented figure eight graph, V_Γ be its vertex and E_Γ^1 and E_Γ^2 be its edges. Set $\bar{S}$ to be a set of all the free homotopy classes of mappings of Γ into F. Denote by $\bar{G}$ the free $\mathbb{Z}$-module generated by $\bar{S}$. Let $K_1 \cup K_2 = L \subset E$ be an oriented two-component link, in general position with respect to p. Note, that local writhe $\omega(q)$ is well defined for a point $q \in p(K_1) \cap p(K_2)$. Let $\bar{G}_q \in \bar{S}$ be the class of the mapping of Γ onto $p(K_1) \cup p(K_2)$, which maps V_Γ to q, E_Γ^1 to $p(K_1)$, E_Γ^2 to $p(K_2)$ (according to the orientations of the edges) and is injective on the complement of the preimage of the double points of $p(L)$. Define $U_L \in \bar{G}$ by the following formula, where the summation is taken over $p(K_1) \cap p(K_2)$

$$U_L = \sum_{q \in p(K_i) \cap p(K_j)} \omega(q) \bar{G}_q \qquad (2.9)$$

2.5.B. THEOREM. *U_L is an isotopy invariant of the link L.*

The proof of Theorem 2.5.B is straightforward. One just has to check, that U_L is invariant under all the oriented versions of the Reidemeister moves.

2.5.C. If $E = \mathbb{R}^3$ and $F = \mathbb{R}^2$, then $\bar{G} = \mathbb{Z}$ (as $\pi_1(\mathbb{R}^2) = e$). Under this identification $U_L = 2\,\mathrm{lk}(K_1, K_2)$, where $\mathrm{lk}(K_1, K_2)$ is the linking number of the two knots.

2.5.D. Let $L = K_1 \cup \cdots \cup K_n \subset E$ be a generic n-component oriented link. For $i > j$ $(i, j \in \{1, \ldots, n\})$ set L_{ij} to be the two component sublink of L, consisting of K_i and K_j. Similarly to Theorem 2.3.C, one can see, that the ordered set of the invariants U_{K_i} and $U_{L_{ij}}$ $(i > j)$ is the most refined degree one Vassiliev invariant of L.

3. TURAEV'S SHADOWS OF KNOTS.

3.1. Preliminary constructions. Let π be an oriented S^1-fibration of N over an oriented closed surface F.

N admits a fixed point free involution, which preserves fibers. Let $\tilde{N}$ be N, factorized by this involution and $p : N \to \tilde{N}$ be the corresponding two-fold covering. Each fiber of p (a pair of antipodal points) is contained in a fiber of π. Therefore, π factorizes into p and a fibration $\tilde{\pi} : \tilde{N} \to F$. Fibers of $\tilde{\pi}$ are projective lines. They are homeomorphic to circles.

An isotopy of a link $L \subset N$ is said to be vertical with respect to π, if each point of L moves along a fiber of π. It is clear, that if two links are vertically isotopic, then their projections are the same. Using vertical isotopy, we modify each generic link L in such a way, that any two points of L, that belong to the same fiber, are in the same orbit of the involution. Denote the generic link obtained, by L'.

Let $\tilde{L} = p(L')$. It is obtained from L' by gluing together points, lying over the same point of F. Hence, $\tilde{\pi}$ maps $\tilde{L}$ bijectively to $\pi(L) = \pi(L')$. Let $r : \pi(L) \to \tilde{L}$ be an inverse bijection. It is a section of $\tilde{\pi}$ over $\pi(L)$.

For a generic non-empty collection of curves on a surface, by a *region* we mean the closure of a connected component of the complement of this collection. Let X be a region for $\pi(L)$ on F, then $\tilde{\pi}\big|_X$ is a trivial fibration. Hence, we can identify it with the projection $S^1 \times X \to X$. Let ϕ be a composition of the section $r\big|_{\partial X}$ with the projection to S^1. It maps ∂X to S^1. Denote by α_X the degree of ϕ (this is actually an obstruction to an extension of $r\big|_{\partial X}$ over X). One can see, that α_X does not depend on the choice of the trivialization of $\tilde{\pi}$ and on the choice of L'.

3.2. Basic definitions and properties.

3.2.A. Definition. The number $\frac{1}{2}\alpha_X$, corresponding to a region X, is said to be the *gleam* of X and is denoted by $\mathrm{gl}(X)$. A *shadow $s(L)$ of a generic link* $L \subset N$ is a (generic) collection of curves $\pi(L) \subset F$ with the gleams, assigned to each region X. The sum of gleams over all the regions is said to be the *total gleam* of the shadow.

3.2.B. One can check, that for any region X the integer α_X is congruent modulo 2 to the number of corners of X. Therefore, $\mathrm{gl}(X)$ is an integer, if the region X has even number of corners and half-integer otherwise.

3.2.C. The total gleam of the shadow is equal to the Euler number of π.

3.2.D. Definition. A *shadow* on F is a generic collection of curves together with the numbers $\mathrm{gl}(X)$ assigned to each region X. These numbers can be either integers or half-integers and they should satisfy the conditions of 3.2.B and 3.2.C.

There are three local moves S_1, S_2 and S_3 of shadows, shown in Figure 4. They are similar to the well known Riedemeister moves of planar knot diagrams.

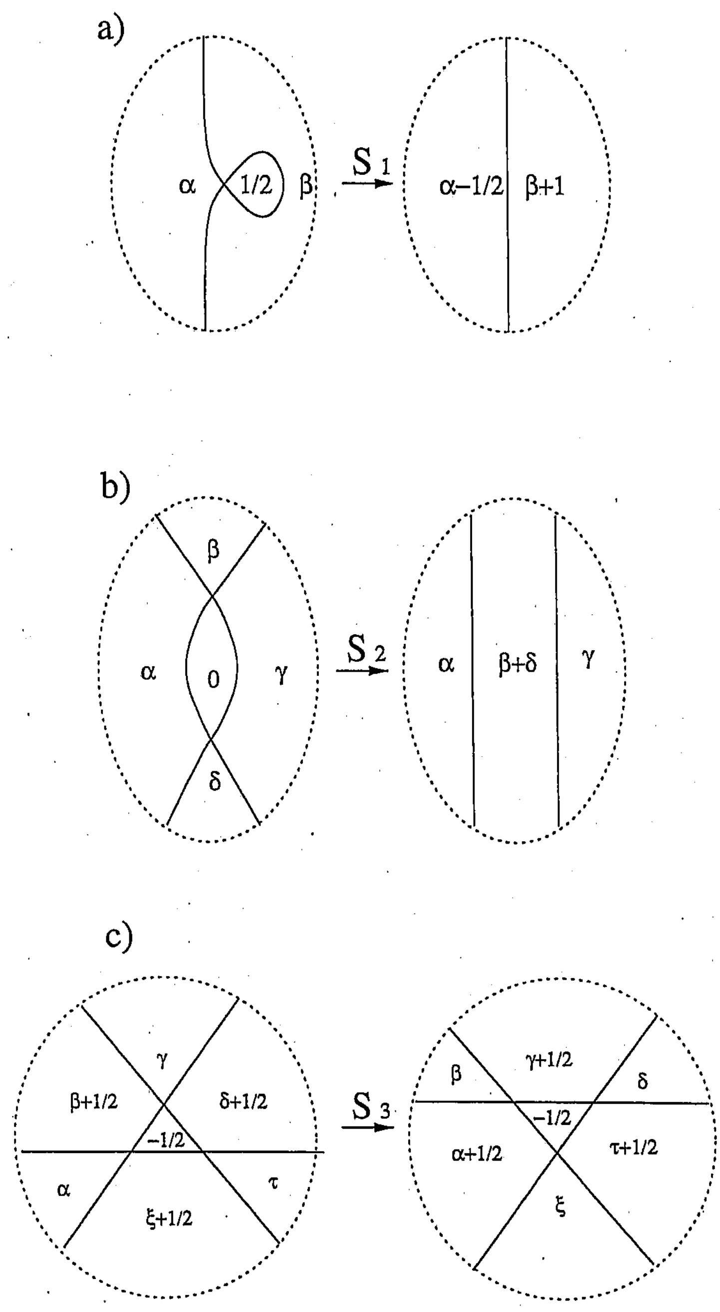

FIGURE 4. Three main shadow moves.

3.2.E. Definition. Two shadows are said to be *(shadow) equivalent*, if they can be transformed to each other by a finite sequence of moves S_1, S_2, S_3 and their inverses.

3.2.F. There are two more important shadow moves $\bar{S}_1$ and $\bar{S}_3$, shown in Figure 5. They are similar to the other versions of the first and the third Riedemeister moves. They can be expressed through S_1, S_2 and S_3 and their inverses.

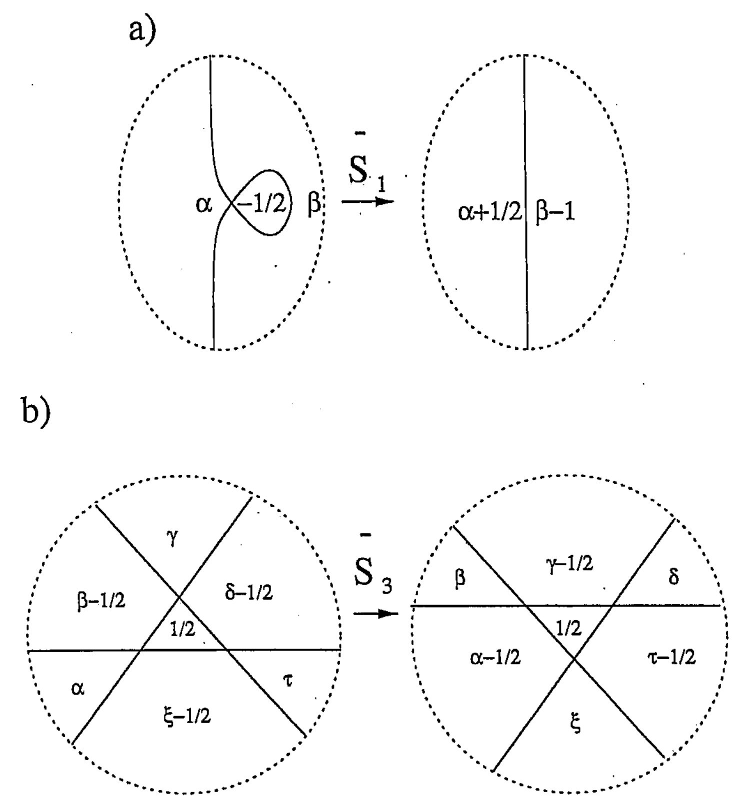

Figure 5. Complimentary shadow moves.

3.2.G. In [9] the action of $H_1(F)$ on the set of all the isotopy types of links in N is constructed as follows.

Let L be a generic link in N and β be an oriented (possibly self intersecting) curve on F, presenting a homology class $[\beta] \in H_1(F)$. Deforming β, we can assume, that β intersects $\pi(K)$ transversally in a finite number of points distinct from the crossing points of $\pi(K)$. Denote by $\alpha = [a, b]$ a small segment of L such, that $\pi(\alpha)$ contains exactly one intersection point c of $\pi(L)$ and β . Assume, that $\pi(a)$ lies to the left, and $\pi(b)$ to the right of β. Replace α by the arc α', shown in Figure 6. We

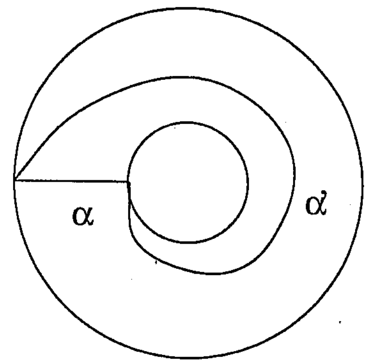

FIGURE 6. Fiber fusion.

call this transformation of L a *fiber-fusion* over the point c. After we apply fiber fusion to L over all the points of $\pi(L) \cap \beta$ we get a new generic link L' with $\pi(L) = \pi(L')$. One could show, that the shadows of K and K' coincide. Indeed, each time when β enters a region X of $s(L)$, it must leave it. Hence, the contributions of the newly inserted arcs to the gleam of X cancel out. Thus, links, belonging to one $H_1(F)$-orbit, always produce the same shadow-link on F.

3.2.H. THEOREM (TURAEV [9]). *Let N be an oriented closed circle bundle over F. The mapping, which associates with each link $L \subset N$ its shadow equivalence class on F, establishes a bijective correspondence between the set of isotopy types of links in N, modulo the action of $H_1(F)$, and the set of all shadow equivalence classes on F with total gleam $\chi(\pi)$. (Here $\chi(\pi)$ is the Euler number of the fibration.)*

3.2.I. It is easy to see, that all the links, such that their projections represent $0 \in H_1(F)$ and their shadows are the same, are homologous to each other. To prove this, one looks at the description of a fiber fusion and notices, that to each fiber fusion, where we add a positive fiber, corresponds one, where we add a negative. Thus the numbers of positively and negatively oriented fibers we add are equal, and they cancel out.

3.2.J. If the surface F is non-orientable and N is oriented, then one can also define a shadow of a knot K in N. To define a gleam of a region X, fix a small disc $D \subset X$. Extend a section $r\big|_{\partial X}$ to a section R of $\tilde{\pi}$ over $X \setminus D$. Take some orientation of D. Together with the orientation of N it defines an oriented longitude l on $T = \tilde{\pi}^{-1}(D)$. The orientation of D induces an orientation on ∂D, and hence on $R\big|_{\partial D}$. There exists $n \in \mathbb{Z}$ such that the oriented curve $R\big|_{\partial D}$ is homologous to nl in

$H_1(T)$. Put $\mathrm{gl}(X) = \frac{n}{2}$. A straightforward check shows that n does not depend on the orientation of D we picked. It is also independent of the choices of D and R. Hence, $\mathrm{gl}(X)$ is well defined.

An immediate check shows, that in the case, when F is oriented, this way of calculating $\mathrm{gl}(X)$ gives the same result as the one, used in Section 3.1.

One immediately generalizes the action of $H_1(F)$, described in 3.2.G, to the case, when F is non-orientable.

3.2.K. In the case of a non-closed surface the gleams of the regions, that have compact closure and do not contain components of boundary, are defined in the same way. To define the gleams of the other regions, we have to fix a section of our fibration over all the boundary components and ends of our surface. In this case the total gleam of the shadow is equal to the the negative obstruction to extension of this section over the whole surface. One can notice, that all the theorems and definitions, stated in this section, can be now passed to the case of a shadow over an arbitrary oriented surface. In Theorem 3.2.H $H_1(F)$ should be substituted by the first homology group with closed support of $\operatorname{Int} F$.

4. Invariants of knots in S^1-fibrations.

4.1. Main constructions. In this section we deal with knots in an S^1-fibration π of an oriented three-dimensional manifold N over an oriented surface F. In this section F and N are not supposed to be closed. As it was said above 3.2.K, all the theorems from the previous section are applicable in this case.

4.1.A. Definition of S_K. The orientations of N and F determine an orientation on a fiber of the fibration. Denote by f the homology class of a positively oriented fiber in $H_1(N)$.

Let $K \subset N$ be an oriented knot, generic with respect to π. Let v be a crossing point of $\pi(K)$. The fiber $\pi^{-1}(v)$ divides K into two halves, which inherit the orientation from K. Complete each half of K by a half of $\pi^{-1}(v)$, such that the orientations on these two arcs define an orientation on their union. The orientations on F and $\pi(K)$ allow one to identify a small neighborhood of v in F with a model picture shown in Figure 7a. Denote the knots, obtained by the operation above, by μ_v^+ and μ_v^-, as it is shown in Figure 7. We will often call this construction a *splitting* of K (with respect to the orientation of K).

This splitting can be described in terms of shadows as follows. Note, that μ_v^+ and μ_v^+ are not in general position. We change them slightly in the neighborhood of $\pi^{-1}(v)$ in such a way, that

$\pi(\mu_v^+)$ and $\pi(\mu_v^+)$ do not have double points in the neighborhood of v. Let P be a neighborhood of v in F, homeomorphic to a closed disc. Fix a section over ∂P, such that the intersection points of $K \cap \pi^{-1}(\partial P)$ belong to the section. Inside P we can construct the Turaev shadow (see 3.2.K). The action of $H_1(\mathrm{Int}\, P) = e$ on the set of the isotopy types of links is trivial (see 3.2.H). Thus, the part of K can be reconstructed in the unique way (up to an isotopy fixed on ∂P) from the shadow over P. The shadows for μ_v^+ and μ_v^- are shown in Figure 7a and Figure 7b, respectively.

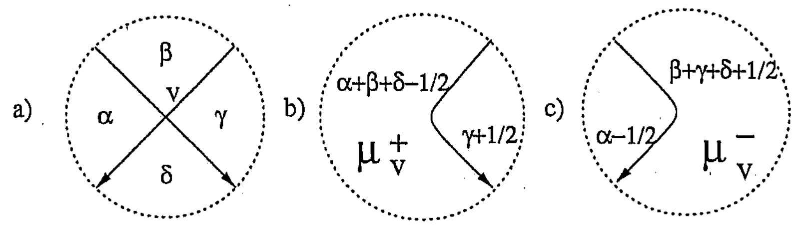

FIGURE 7. Shadows for the splitting of K.

Regions of the shadows $s(\mu_v^+)$ and $s(\mu_v^-)$ are, in fact, unions of regions for $s(K)$. One should think of gleams as of measure, thus the gleam of a region is the sum of all the numbers inside.

Let H be the (integer) group ring of $H_1(N)$, factorized (as a $\mathbb{Z}$-module) by the submodule, generated by $\{[K] - f, [K]f - e\}$. Here by $[K] \in H_1(N)$ we mean the homology class represented by the image of K.

At last define $S_K \in H$ by the following formula, where the summation is taken over all the crossing points v of $\pi(K)$

$$S_K = \sum_v ([\mu_v^+] - [\mu_v^-]) \tag{4.1}$$

4.1.B. Since $\mu_v^+ \cup \mu_v^- = K \cup \pi^{-1}(v)$ we get, that

$$[\mu_v^+][\mu_v^-] = [K]f \tag{4.2}$$

4.1.C. THEOREM. *S_K is an isotopy invariant of the knot K.*

For the proof of Theorem 4.1.C see Section 8.4.

4.1.D. From 4.1.B it follows, that S_K can be also described as an element of $\mathbb{Z}[H_1(N)]$, equal to a sum of $([\mu_v^+] - [\mu_v^-])$ over all the double points for which the sets $\{[\mu_v^+], [\mu_v^-]\}$ and $\{e, f\}$ are disjoint. Note, that in this case we do not need to factorize $\mathbb{Z}[H_1(N)]$, to make S_K well defined.

4.1.E. One can obtain an invariant similar to S_K, taking values in the free $\mathbb{Z}$-module generated by the set of all the free homotopy classes of oriented curves in N. To do this, one substitutes the homology classes of μ_v^+ and μ_v^- in (4.1) by their free homotopy classes and takes the summation over the set of all the crossing points v of $\pi(K)$, such that neither one of μ_v^+ and μ_v^- is homotopic to a trivial loop and neither one of them is homotopic to a positively oriented fiber (see 4.1.D).

To prove, that this is really an invariant of K one can easily modify the proof of Theorem 4.1.C.

4.2. S_K is a Vassiliev invariant of degree one.

4.2.A. If a fiber-fusion changes the gleam γ in Figure 7b by $+1$, then $[\mu_v^+]$ changes by multiplication by f. If a fiber-fusion changes the gleam α in Figure 7c by $+1$, then $[\mu_v^-]$ changes by multiplication by f^{-1}. These facts are easy to check.

4.2.B. Let us see, how S_K changes under the modification (see 2.2.D) along a fiber over a crossing point v. Consider a singular knot K', (whose only singularity is a point v of transverse self-intersection). Let ξ_1 and ξ_2 be the homology classes of the two loops of K' adjacent to v. The two resolutions of this double point correspond to adding $\pm\frac{1}{2}$ to the gleams of the regions adjacent to v in the two ways, shown in Figure 8b and Figure 8c.

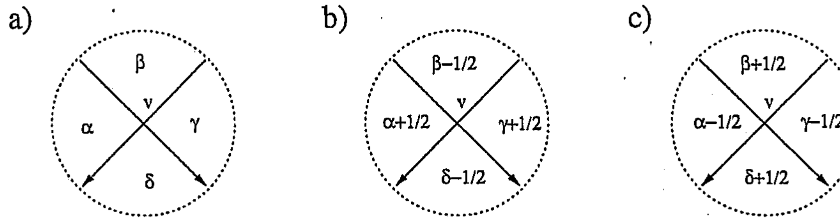

FIGURE 8. Shadow description of fiber modification.

Using 4.2.A one checks, that under the corresponding modification S_K changes by

$$(f - e)(\xi_1 + \xi_2) \tag{4.3}$$

This means, that the first derivative of S_K depends only on the homology classes of the two loops adjacent to the singular point. Hence, the second derivative of S_K is 0. Thus, it is a Vassiliev invariant of degree one in the usual sense.

Because of the similar reasons, the version of S_K, having values in the free $\mathbb{Z}$-module generated by all the free homotopy classes of oriented curves in N, is also a Vassiliev invariant of degree one.

4.2.C. THEOREM. I: *If K and K' are two knots, representing the same free homotopy class, then S_K and $S_{K'}$ are congruent modulo the submodule generated by elements of form*

$$(f-e)(j+[K]j^{-1}) \tag{4.4}$$

(for $j \in H_1(N)$).

II: *If K is a knot, and $S \in H$ is congruent to S_K modulo the submodule generated by elements of form* (4.4) *(for $j \in H_1(N)$), then there exists a knot K', such that:*

a) K and K' represent the same free homotopy class

b) $S_{K'} = S$

For the proof of Theorem 4.2.C see Section 8.5.

4.2.D. Fiber-wise compactification of an $\mathbb{R}^1$-fibration is an S^1-fibration. Thus for an oriented knot K in an oriented $\mathbb{R}^1$-fibration E over an oriented surface F both the U_K and the S_K invariants are defined. There is a formula relating the versions of them taking values in the group rings of the homology groups (see 4.1.D and 2.2.C). In order to state it we introduce the following notation.

Fix an arbitrary knot in each free homotopy class of a curve in E. Denote by $\tilde{K}$ this fixed knot in the free homotopy class realized by K. Put $\Delta U_K = U_K - U_{\tilde{K}}$, and $\Delta S_K = S_K - S_{\tilde{K}}$.

4.2.E. THEOREM. $\frac{1}{2}\Delta U_K(f-e) = \Delta S_K$

For the proof of Theorem 4.2.E see Section 8.7.

4.2.F. COROLLARY. *The values of $U_{\tilde{K}}$, U_K and $S_{\tilde{K}}$ invariants, determine S_K.*

4.2.G. DEFINITION OF S_C. Let C be a generic oriented curve on an oriented surface F. Let H be the free $\mathbb{Z}$-module generated by free homotopy classes of curves on F. For a double point v set $\mu^+(v)$ and $\mu^-(v)$ be the two curves obtained by splitting of C at v according to the orientation (see Figure 7, where they correspond to μ_v^+ and μ_v^-). As before we can distinguish them (see 4.1.A). Set $\mu_h^+(v)$ and $\mu_h^-(v)$ to be the free homotopy classes realized by them. Define $S_C \in H$ by the following formula where the summation is made over the set of all the double points of C for which none of $\mu^+(v)$ and $\mu^-(v)$ is homotopic to a trivial loop.

$$S_C = \sum_v (\mu_h^+(v) - \mu_h^-(v)) \tag{4.5}$$

4.2.H. THEOREM. *S_C is a homotopy invariant of the curve C*

The proof is straightforward. One just shows, that S_C is invariant under the moves of passing through self-tangency point, passing through a triple point and a small kink birth move.

This invariant is closely related to the quadratic form on a certain quotient of $\mathbb{Z}[\pi_1(F)]$ introduced by V. Turaev [10] and explored by V. Turaev and O. Viro [11].

4.3. Example. If N is just a solid torus, fibered over a disc, then we can calculate the value of S_K directly from the shadow of K.

4.3.A. DEFINITION. Let C be an oriented closed curve in $\mathbb{R}^2$, X be a region for C (i.e., it is a closure of a connected component of $\mathbb{R}^2 \setminus C$). Take a point $x \in \operatorname{Int} X$ and connect it to a point near infinity by a generic oriented path D. Define the sign of an intersection point of C and D in the way, shown in Figure 9. Put $\operatorname{ind}_C X$ to be the sum over all the intersection points of C and D of the signs of these points.

FIGURE 9. Sign of a crossing point.

It is easy to see, that $\operatorname{ind}_C(X)$ is independent on the choices of x and D.

4.3.B. DEFINITION. Let $K \subset T$ be an oriented knot, generic with respect to π, and let $s(K)$ be its shadow. Define $\sigma(s(K)) \in \mathbb{Z}$ by the following sum over all the regions X for $\pi(K)$

$$\sigma(s(K)) = \sum_X \operatorname{ind}_{\pi(K)}(X)\, \operatorname{gl}(X) \tag{4.6}$$

Denote by $h \in \mathbb{Z}$ the image of $[K]$ under the natural identification of $H_1(T)$ with $\mathbb{Z}$.

4.3.C. LEMMA. $\sigma(s(K)) = h$

4.3.D. Put

$$S'_K = \sum t^{\sigma(s(\mu_v^+))} - t^{\sigma(s(\mu_v^-))} \tag{4.7}$$

where the sum is taken over all the double points v of $\pi(K)$, such that $\{0,1\}$ and $\{\sigma(s(\mu_v^+)), \sigma(s(\mu_v^-))\}$ are disjoint (see 4.1.D).

Lemma 4.3.C implies, that S'_K is the image of S_K under the natural identification between $\mathbb{Z}[H_1(T)]$ and the ring of finite Laurent polynomials.

One can show, that S'_K and F. Aicardi's partial linking polynomial of K, introduced in [2], can be explicitly expressed through each other.

4.3.E. To relate $S'K$ and $A(K)$, we introduce the following notations. Put

$$A^h = \begin{cases} (t^1 + t^2 + \cdots + t^{h-1}), & \text{if } h > 0 \\ (t^{-1} + t^{-2} + \cdots + t^{h+1}), & \text{if } h < 0 \\ 0, & \text{if } h = 0 \end{cases} \tag{4.8}$$

For $l \in \mathbb{Z}$ denote by α_l the coefficient of t^l in $A(K) - A^h$. (Recall, that $h = [K]$)

4.3.F. Theorem. *Let $K \subset T$ be an oriented knot. Then*

$$S'(K) = (t-1) \times \begin{cases} \Big(A(K) - A^h(1 + \frac{1}{2}\alpha_1 - \frac{1}{2}\alpha_{-1}) \\ \qquad + \frac{1}{2}\alpha_{-1}(t^h + 1)\Big) & \textit{if } h > 2 \\ \Big(A(K) - A^h(1 + \frac{1}{2}\alpha_{-1} - \frac{1}{2}\alpha_1) \\ \qquad + \frac{1}{2}\alpha_1(t^h + 1)\Big) & \textit{if } h < -2 \\ A(K) & \textit{if } h = \pm 1 \\ A(K) - t^{-1}(1 + \frac{1}{2}\alpha_{-1}) & \textit{if } h = -2 \\ A(K) - t^1(1 + \frac{1}{2}\alpha_1) & \textit{if } h = 2 \\ A(K) + \frac{1}{2}\alpha_1 & \textit{if } h = 0 \end{cases} \tag{4.9}$$

4.3.G. Corollary. *Given $[K]$ there exist formulas, that express $S'(K)$ through $A(K)$ and $A(K)$ through $S'(K)$.*

The proof is straightforward.

4.4. Further generalizations of the S_K invariant. One can show, that an invariant similar to S_K can be introduced in the case, when N is oriented and F is non-orientable.

4.4.A. Definition of $\tilde{S}_K$. Let N be oriented and F be non-orientable. Let $K \subset N$ be an oriented knot, generic with respect to π and v be a crossing point of $\pi(K)$. Fix some orientation on a small neighborhood of v on F. As N is oriented, this induces an orientation on the fiber $\pi^{-1}(v)$. Similarly to the definition of S_K (see 4.1.A), we split our knot with respect to the orientation and obtain two knots $\{\mu_1^+(v), \mu_1^-(v)\}$. Then we take the other orientation on the neighborhood of v in F, and in the same way we obtain another pair of knots $\{\mu_2^+(v), \mu_2^-(v)\}$. The element $([\mu_1^+(v)] - [\mu_1^-(v)] + [\mu_2^+(v)] - [\mu_2^-(v)]) \in \mathbb{Z}[H_1(N)]$ does not depend on, which orientation on the neighborhood of v we choose first.

Similarly to the definition of S_K, we can describe all this in terms of shadows, as it is shown in Figure 10. These shadows are constructed with respect to the same orientation on the neighborhood of v.

Let f be the homology class of a fiber of π, oriented in some way. As one can easily prove $f^2 = e$, so it does not matter, which orientation we choose to define f. Let $\tilde{H}$ be $\mathbb{Z}[H_1(N)]$ factorized (as a $\mathbb{Z}$-module) by the $\mathbb{Z}$- submodule generated by $\Big\{[K] - f + e - [K]f = (e - f)([K] + e)\Big\}$. At last define $\tilde{S}_K \in \tilde{H}$ by the following formula, where the summation is taken over all the crossing points v of $\pi(K)$.

$$\tilde{S}_K = \sum_v \Big([\mu_1^+(v)] - [\mu_1^-(v)] + [\mu_2^+(v)] - [\mu_2^-(v)]\Big) \tag{4.10}$$

4.4.B. Theorem. *$\tilde{S}_K$ is an isotopy invariant of the knot K.*

The proof is essentially the same as the proof of Theorem 4.1.C.

4.4.C. One can easily prove, that $\tilde{S}_K$ invariant satisfies relations similar to (4.3). In particular $\tilde{S}_K$ is also a Vassiliev invariant of degree one. The analogue of Theorem 4.2.C also holds for $\tilde{S}_K$.

One can introduce a version of this invariant, taking values in the free $\mathbb{Z}$-module generated by all the free homotopy classes of oriented curves in N. To do this, we substitute the homology classes of $\mu_1^+(v)$, $\mu_1^-(v)$, $\mu_2^+(v)$ and $\mu_2^-(v)$ by the free homotopy classes realized by them. The summation should be taken over the set of all the double points of $\pi(K)$, for which neither one of $\mu_1^+(v), \mu_1^-(v), \mu_2^+(v)$ and $\mu_2^-(v)$ is homotopic to a trivial loop and neither one of them is homotopic to a fiber of π. To prove, that this is really an invariant of K, one easily modifies the proof of Theorem 4.1.C.

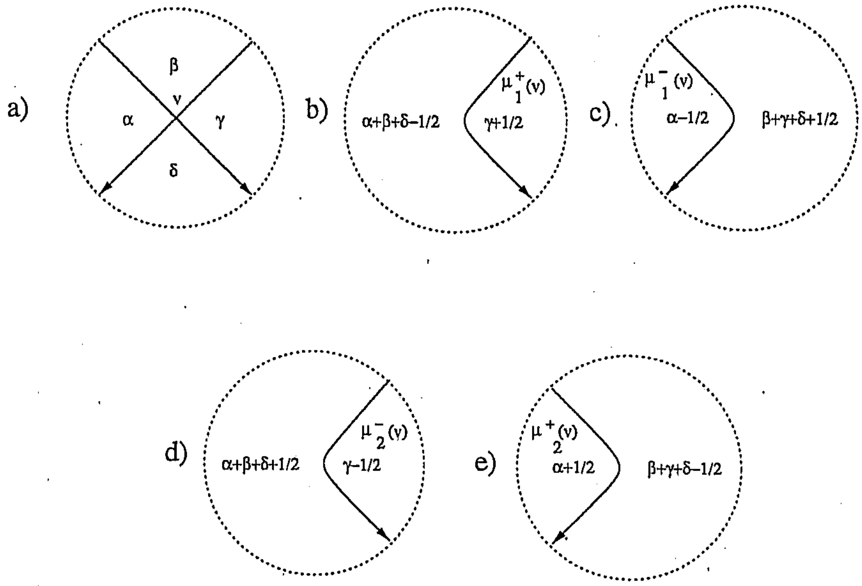

FIGURE 10. S_K in the case of nonorientable surface.

5. Invariants of knots in Seifert fibered spaces

Let (μ, ν) be a pair of relatively prime integers. Let

$$D^2 = \left\{(r,\theta); 0 \le r \le 1, 0 \le \theta \le 2\pi\right\} \subset \mathbb{R}^2 \tag{5.1}$$

be the unit disc, defined in polar coordinates. A fibered solid torus of type (μ, ν) is the quotient space of the cylinder $D^2 \times I$ via the identification $\left((r,\theta),1\right) = \left((r,\theta+\frac{2\pi\nu}{\mu}),0\right)$. The fibers are the images of the curves $x \times I$. The number μ is called the index or the multiplicity. If $|\mu| > 1$ the fibered solid torus is said to be *exceptionally fibered* and the fiber, which is the image of $0 \times I$, is called an *exceptional fiber*. Otherwise the fibered solid torus is said to be *regularly fibered* and each fiber is a regular fiber.

5.0.D. Definition. An orientable three manifold S is said to be a *Seifert fibered manifold*, if it is a union of pairwise disjoint closed curves, called fibers, such that each one has a closed neighborhood consisting of a union of fibers, which is homeomorphic to a fibered solid torus via a fiber preserving homeomorphism.

A fiber, h is called *exceptional*, if it has a neighborhood homeomorphic to an exceptionally fibered solid torus (via a fiber preserving homeomorphism) and h corresponds, via the homeomorphism to the exceptional fiber of the solid torus. If $\partial S \neq 0$ then ∂S should be a union of regular fibers.

The quotient space, obtained from a Seifert fibered manifold S by identifying each fiber to a point, is a 2-manifold. We call it the orbit space. The images of the exceptional fibers are called *the cone points.*

5.0.E. For an exceptional fiber a of an oriented Seifert fibered manifold there is a unique pair of relative prime integers (μ_a, ν_a), such that $\mu_a > 0$, $|\nu_a| < \mu_a$ and a neighborhood of a is homeomorphic, via a fiber preserving homeomorphism to a fibered solid torus of type (μ_a, ν_a). We call the pair (μ_a, ν_a) *the type of the exceptional fiber a.* We also call this pair the type of the corresponding cone point.

We can define an invariant similar to the S_K invariant of oriented knots in a Seifert fibered manifold.

Clearly, any S^1-fibration can be viewed as a Seifert fibration without cone points. This justifies the notation in the definition below.

5.0.F. Definition of S_K. Let N be an oriented Seifert fibered manifold with an oriented orbit space F. Let $\pi : N \to F$ be the corresponding fibration, and $K \subset N$ be an oriented knot in general position with respect to π. Assume also, that K does not intersect the exceptional fibers. For each double point v of $\pi(K)$ we split K into μ_v^+ and μ_v^- as in 4.1.A. Let A be the set of all the exceptional fibers. Since N and F are oriented we have an induced orientation on each exceptional fiber $a \in A$. For $a \in A$ set f_a to be the homology class of the fiber with this orientation. For $a \in A$ of type (μ_a, ν_a) (see 5.0.E) set $N_1(a) = \{k \in \{1, \dots, \mu_a\} | \frac{2\pi k \nu_a}{\mu_a} \, mod \, 2\pi \in (0, \pi]\}$, $N_2(a) = \{k \in \{1, \dots, \mu_a\} | \frac{2\pi k \nu_a}{\mu_a} \, mod \, 2\pi \in (0, \pi)\}$. Define $R_a^1, R_a^2 \in \mathbb{Z}[H_1(N)]$ by the following formulas:

$$R_a^1 = \sum_{k \in N_1(a)} ([K] f_a^{\mu_a - k} - f_a^k) - \sum_{k \in N_2(a)} (f_a^{\mu_a - k} - [K] f_a^k) \tag{5.2}$$

$$R_a^2 = \sum_{k \in N_1(a)} (f_a^{\mu_a - k} - [K] f_a^k) - \sum_{k \in N_2(a)} ([K] f_a^{\mu_a - k} - f_a^k). \tag{5.3}$$

Let H be $\mathbb{Z}[(H_1(N)]$ factorized (as a $\mathbb{Z}$-module) by the free $\mathbb{Z}$-submodule generated by $\Big\{[K]f - e, [K] - f, \{R_a^1, R_a^2\}_{a \in A}\Big\}$. Finally, define $S_K \in H$ by the following formula, where the summation is taken over all the double points v of $\pi(K)$

$$S_K = \sum_v ([\mu_v^+] - [\mu_v^-]) \tag{5.4}$$

5.0.G. THEOREM. *S_K is an isotopy invariant of the knot K.*

For the proof of Theorem 5.0.G see Section 5.0.G.

We introduce a similar invariant in the case, when N is oriented and F is non-orientable.

5.0.H. DEFINITION OF $\tilde{S}_K$. Let N be an oriented Seifert fibered manifold with a non-orientable orbit space F. Let $\pi : N \to F$ be the corresponding fibration, and $K \subset N$ be an oriented knot in general position with respect to π. Assume also, that K does not intersect the exceptional fibers. For each double point v of $\pi(K)$ we split K into $\mu_1^+(v)$, $\mu_1^-(v)$, $\mu_2^+(v)$ and $\mu_2^-(v)$ as in 4.4.A. The element $([\mu_1^+(v)] - [\mu_1^-(v)] + [\mu_2^+(v)] - [\mu_2^-(v)]) \in \mathbb{Z}[H_1(N)]$ is well defined.

Denote by f the homology class of a regular fiber oriented in some way. Note that $f^2 = e$, so the orientation we use to define f does not matter. For a cone point a denote by f_a the homology class of the fiber $\pi^{-1}(a)$ oriented in some way.

For $a \in A$ of type (μ_a, ν_a) set $N_1(a) = \{k \in \{1, \dots, \mu_a\} | \frac{2\pi k \nu_a}{\mu_a}\ mod\ 2\pi \in (0, \pi]\}$ $N_2(a) = \{k \in \{1, \dots, \mu_a\} | \frac{2\pi k \nu_a}{\mu_a}\ mod\ 2\pi \in (0, \pi)\}$

Define $R_a \in \mathbb{Z}[H_1(N)]$ by the following formula:

$$R_a = \sum_{k \in N_1(a)} \left([K] f_a^{\mu_a - k} - f_a^k + f_a^{k-\mu_a} - [K] f_a^{-k} \right) - \sum_{k \in N_2(a)} \left(f_a^{\mu_a - k} - [K] f_a^k + [K] f_a^{k-\mu_a} - f_a^{-k} \right) \tag{5.5}$$

Put $\tilde{H}$ to be $\mathbb{Z}[H_1(N)]$, factorized (as a $\mathbb{Z}$-module) by the free $\mathbb{Z}$-submodule generated by $\Big\{(e - f)([K] + e), \{R_a\}_{a \in A}\Big\}$.

One can prove, that under the change of the orientation on $\pi^{-1}(a)$ (used to define f_a) R_a goes to $-R_a$. Thus $\tilde{H}$ is well defined. To show this, one checks that, if μ_a is odd, then $N_1(a) = N_2(a)$. Under this change each term from the first sum (used to define R_a) goes to minus the corresponding term from the second sum and vice versa. (Note that $f^2 = e$.) If μ_a is even and $\mu_a = 2l$ (for some $l \in \mathbb{Z}$), then $N_1(a) \setminus \{l\} = N_2(a)$. Under this change each term with $k \in N_1(a) \setminus \{l\}$ goes to minus the corresponding term with $k \in N_2(a)$ and vice versa. The term in the first sum, which corresponds to $k = l$, goes to minus itself.

Finally define $\tilde{S}_K \in \tilde{H}$ by the sum over all the double points v of $\pi(K)$

$$\tilde{S}_K = \sum_v \Big([\mu_1^+(v)] - [\mu_1^-(v)] + [\mu_2^+(v)] - [\mu_2^-(v)] \Big) \tag{5.6}$$

5.0.I. THEOREM. *$\tilde{S}_K$ is an isotopy invariant of K.*

The proof is a straightforward generalization of the proof of Theorem 5.0.G.

5.0.J. There is a version of S_K, taking values in the free $\mathbb{Z}$-module generated by all the free homotopy classes of oriented curves in N. To obtain it, one substitutes all the homology classes in the definitions by the corresponding free homotopy classes. The summation should be made over all the double points of $\pi(K)$, for which neither one of the knots, obtained by the splitting, is homotopic to a trivial loop and neither one of them is homotopic to a positively oriented fiber of N. Each homology class of the form f_a^k in the definitions of R_a^1 and R_a^2 should be substituted by the free homotopy class of the curve, which goes along the exceptional fiber k times. Each class of the form $[K]f_a^k$ should be substituted by the free homotopy class of K with a curve, going along the exceptional fiber k times, added to it. (Note, that f_a is in the center of $\pi_1(N)$, so this class is well defined.) To prove, that this is really an invariant of K, one easily modifies the proof of Theorem 5.0.G.

The version of $\tilde{S}_K$, taking values in the free $\mathbb{Z}$-module, generated by the set of all the free homotopy classes of oriented curves in N, is constructed in a similar way.

5.0.K. One can easily check, that S_K, $\tilde{S}_K$ and their versions described in 5.0.J satisfy relations similar to (4.3)). Hence all of them are Vassiliev invariants of degree one (see 4.2.B). The corresponding versions of Theorem 4.2.C also hold for them.

6. WAVE FRONTS ON SURFACES

6.1. Definitions. Let F be a two-dimensional manifold. A *contact element* at a point on F is a one-dimensional vector subspace in the tangent plane. This subspace divides the tangent plane into two halves. A choice of one of them is called a *coorientation* of a contact element. The space of all the cooriented contact elements of F is a spherization of the cotangent bundle of F, that is ST^*F. We will also denote it by N. It is an S^1-fibration over F. A *Legendrian* curve λ in N is an immersion of S^1 into N, such that the tangent vector to λ at each point lies in the contact plane. The projection $L \subset F$ of a cooriented Legendrian curve $\lambda \subset N$ is called the *the front* of λ. A wave front is said to be generic, if it is an immersion everywhere except a finite number of points, where it has cusp singularities, and all the multiple points are double points of transversal self-intersection. A cusp is the projection of a point, where the corresponding Legendrian curve is tangent to the fiber of the bundle.

Cooriented generic front may be uniquely lifted to a Legendrian curve $\lambda \subset N$, by taking a coorienting normal direction as a contact element at each point of the front.

6.2. Shadows of wave fronts.

6.2.A. For any surface F, the space ST^*F is canonically oriented. The orientation is constructed as follows. For a point $x \in F$ fix an orientation on T_xF. It induces an orientation on the fiber over x. These two orientations determine an orientation on three dimensional planes tangent to the points of fiber over x. A straightforward check shows, that this orientation is independent on the orientation on T_xF we choose. Hence, the orientation on ST^*F is well defined.

Thus, for a generic knot in ST^*F the notion of shadow is well defined (see 3.1.I and 3.2.J). Theorem 6.2.C describes the shadow of a Legendrian lifting of a generic cooriented wave front $L \subset F$.

6.2.B. Definition. Let X be a connected component of $F \setminus L$. We denote by C^i_X the number of cusps in the boundary of the region X pointing inside X (as in Figure 11a), by C^o_X the number of cusps in the boundary of X pointing outside (as in Figure 11b), and by V_X the number of corners of X, where the picture locally looks in one of the two ways, shown in Figure 11c. It can happen, that a cusp point is pointing both inside and outside of X. In this case it makes input both into C^i_X and C^o_X. If the corner of the type shown in Figure 11c participates two times in ∂X, then it should be counted twice, to get the value of V_X.

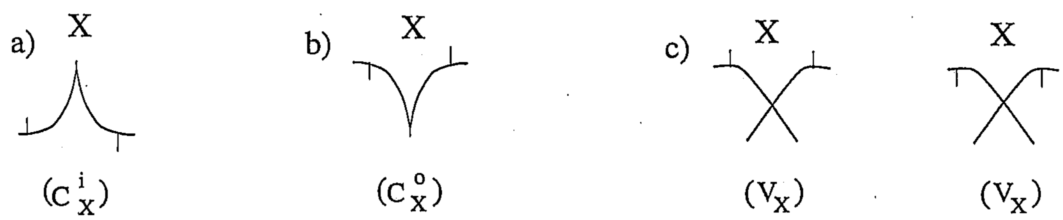

Figure 11. Types of crossing and cusp points.

6.2.C. Theorem. *Let L be a generic cooriented wave front on a surface F (not necessarily closed or compact), corresponding to a Legendrian curve λ. There exists a small deformation of λ, in the* *class of all smooth (not only Legendrian) curves such, that its result is a curve generic with respect to the projection, and the shadow of this curve can be constructed in the following way. We replace a small neighborhood of each cusp of L by a smooth simple arc. The gleam of an arbitrary region X, which has compact closure and does not contain boundary components of F, is calculated via the following formula:*

$$\mathrm{gl}_X = \chi\,\mathrm{Int}(X) + \frac{1}{2}(C^i_X - C^o_X - V_X) \tag{6.1}$$

(Here χ denotes the Euler characteristic.)

For the proof of Theorem 6.2.C see Section 8.10.

Note, that as it was said in 3.2.K, the gleam of a region, that does not have compact closure or contains boundary components is not defined.

6.2.D. A self-tangency point p of a wave front is said to be a point of *dangerous self-tangency*, if the coorienting normals of the two branches coincide at p (see Figure 12). A dangerous self-tangency point corresponds to a self-intersection of the Legendrian curve. Hence, a generic deformation of the front L, not involving *dangerous* self tangencies, corresponds to an isotopy of the Legendrian knot λ.

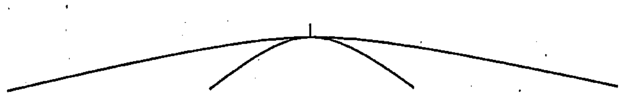

FIGURE 12. Dangerous selftangency.

Any generic deformation of a wave front L, which corresponds to an isotopy in the class of the Legendrian knots, can be splitted into a sequence of modifications, depicted in Figure 13. The construction of Theorem 6.2.C transforms these generic modifications of the wave front to shadow moves: Ia and Ib in Figure 13 are transformed to the $\bar{S}_1$ move for shadow diagrams, IIa, IIb, II'a, II'b, II'c and II'd are transformed to the S_2 move, finally IIIa and IIIb are transformed to S_3 and $\bar{S}_3$, respectively.

6.2.E. Thus, we are able to calculate for the Legendrian lifting of a wave front all the invariants, which we can calculate for shadows. This includes the analogue of the linking number for the fronts on $\mathbb{R}^2$ (see [9]), the second degree Vassiliev invariant (see A. Shumakovitch [6]) and quantum state sums (see [9]).

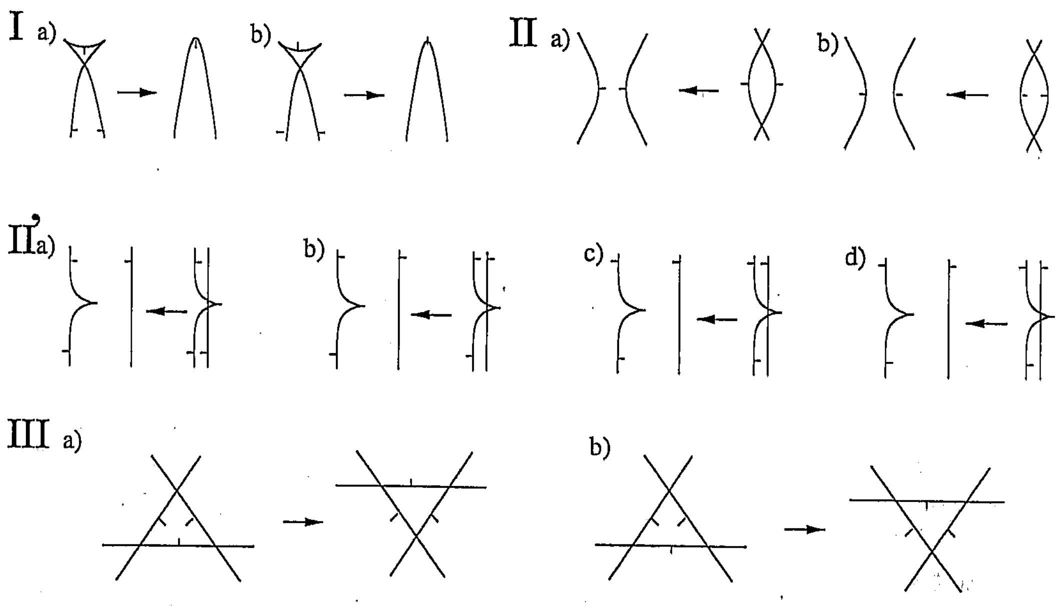

FIGURE 13. Wave front moves.

6.3. Invariants of wave fronts on surfaces. In particular, the S_K invariant gives rise to an invariant of a generic wave front. This invariant appears to be related to the formula for the Bennequin invariant of a wave front, introduced by M. Polyak in [5].

Recall the corresponding results and definitions of [5].

Let L be a generic cooriented oriented wave front on an oriented surface F. A branch of a wave front is said to be positive (resp. negative), if the frame of coorienting and orienting vectors defines positive (resp. negative) orientation on the surface F. Define the *sign* $\epsilon(v)$ of the crossing point v of L to be $+1$, if signs of both branches of the curve, intersecting at v, coincide and -1, otherwise. Similarly, we assign a positive (negative) sign to a cusp point, if the coorienting vector turns in a positive (negative) direction, while traversing a small neighborhood of the cusp point along the orientation. We denote half of the number of positive and negative cusp points by C^+ and C^-, respectively.

Let v be a crossing point of L. The orientations of F and L together with the coorientation of L allow one to identify a small neighborhood of v in F with one of the model pictures shown in Figure 14a and Figure 14b. Denote by L_v^+ and L_v^- the wave fronts, obtained by splitting of L in v according to orientation and coorientation, as it is shown in Figure 14a.1) and Figure 14b.1).

For a Legendrian curve λ in $ST^*\mathbb{R}^2$ denote by $l(\lambda)$ the Bennequin invariant of it, described in the

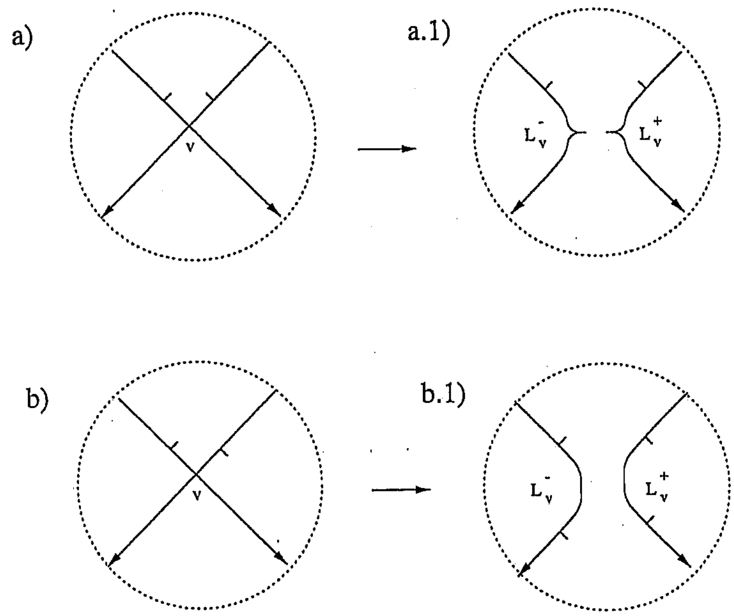

FIGURE 14. Symmetric splitting of a wave front.

works of S. Tabachnikov [7] and V. Arnold [3] with the sign convention of [3] and [5].

6.3.A. THEOREM. *([Polyak [5]]) Consider a generic oriented cooriented wave front L on $\mathbb{R}^2$. Let λ be the corresponding Legendrian curve. Denote by $\operatorname{ind}(L)$ the degree of the mapping, taking a point p on the front to the point on S^1, where the coorienting normal at p points to. Define S as the sum over all the crossing points of L:*

$$S = \sum_v (\operatorname{ind}(L_v^+) - \operatorname{ind}(L_v^-) - \epsilon(v)) \tag{6.2}$$

Then

$$l(\lambda) = S + (1 - \operatorname{ind}(L))C^+ + (\operatorname{ind}(L) + 1)C^- + \operatorname{ind}(L)^2 \tag{6.3}$$

As it was shown in [5], the Bennequin invariant of a wave front on the $\mathbb{R}^2$ plane admits quantization. Consider a formal quantum parameter q. Recall, that for any $n \in \mathbb{Z}$ the corresponding quantum number $[n]_q \in \mathbb{Z}[q, q^{-1}]$ is a finite Laurent polynomial in q, defined by $[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$. Substituting quantum integers instead of integers in 6.3.A, we get the following theorem.

6.3.B. THEOREM. *([Polyak [5]]) Let L be a generic cooriented oriented wave front on $\mathbb{R}^2$ and λ be the corresponding Legendrian curve. Define S_q by the following formula, where the sum is taken* *over the set of all the double points of L.*

$$S_q = \sum_v [ind(L_v^+) - ind(L_v^-) - \epsilon(v)]_q \tag{6.4}$$

Put

$$l_q(L) = S_q + [1 - ind(L)]_q C^+ + [ind(L) + 1]_q C^- + [ind(L)]_q ind(L) \tag{6.5}$$

Then $l_q(\lambda) = l_q(L) \in \frac{1}{2}\mathbb{Z}[q, q^{-1}]$ is invariant under isotopy in the class of the Legendrian knots.

The $l_q(\lambda)$ invariant can be expressed [1] through the partial linking polynomial of a generic cooriented oriented wave front, introduced by F. Aicardi [2].

The reason why this invariant takes values in $\frac{1}{2}\mathbb{Z}[q, q^{-1}]$ and not in $\mathbb{Z}[q, q^{-1}]$ is that the number of positive (or negative) cusps can be odd. This makes C^+ (C^-) a half-integer.

Let λ_v^ϵ with $\epsilon = \pm$ be the Legendrian lifting of the front L_v^ϵ. Let $f \in H_1(ST^*F)$ be the homology class of a positively oriented fiber.

6.3.C. Theorem. *([Polyak [5]]) Let L be a generic cooriented wave oriented front on an oriented surface F. Let λ be the corresponding Legendrian curve. Define $l_F(\lambda) \in H_1(ST^*F, \frac{1}{2}\mathbb{Z})$ by the following formula.*

$$l_F(\lambda) = \sum_v \Big([\lambda_v^+] - [\lambda_v^-] - \epsilon(v)f\Big) + (f - [\lambda])C^+ + ([\lambda] + f)C^- \tag{6.6}$$

(Here we use an additive notation for the group operation in $H_1(ST^*F, \frac{1}{2}\mathbb{Z})$.)

Then $l_F(\lambda)$ is invariant under isotopy in the class of the Legendrian knots.

The proof is straightforward. One checks that $l_F(\lambda)$ is really invariant under all the oriented versions of non-dangerous self-tangency, triple point, cusp crossing and cusp birth moves of the wave front.

In [5] this invariant is denoted by $I_\Sigma^+(\lambda)$ and, in a sense, it appears to be a natural generalization of the Arnold's J^+ invariant [3], to the case of an oriented cooriented wave front on an oriented surface.

Note, that in the situation of Theorem 6.3.A the indices of all the fronts, involved, are the images of the homology classes of their Legendrian liftings under the natural identification between $H_1(ST^*\mathbb{R}^2)$ and $\mathbb{Z}$. If one replaces everywhere in 6.3.A indices by the corresponding homology classes and puts f instead of 1 then the only difference between the two formulas is the term $\text{ind}^2(L)$.

6.3.D. The splitting of the knot K into μ_v^+ and μ_v^-, used to define S_K (see 4.1.A), in the case when K is a Legendrian lifting of a generic wave front L, can be done up to an isotopy in the class of the Legendrian knots. Although this can be done in many ways, there exists the simplest way. The projections $\tilde{L}_v^+$ and $\tilde{L}_v^-$ of the Legendrian curves, created by the splitting, are shown in Figure 15. (This fact follows from Theorem 6.2.C.)

Let $\tilde{\lambda}_v^\epsilon$ with $\epsilon = \pm$ be the Legendrian lifting of the front $\tilde{L}_v^\epsilon$.

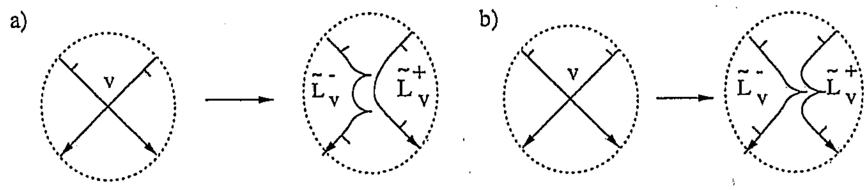

FIGURE 15. Nonsymmetric splitting of a wave front.

6.3.E. THEOREM. *Let L be a generic cooriented wave front on an oriented surface F. Let λ be the corresponding Legendrian curve. Define $S(\lambda) \in \frac{1}{2}\mathbb{Z}\left[H_1(ST^*F)\right]$ by the following formula.*

$$S(\lambda) = \sum_v \left([\tilde{\lambda}_v^+] - [\tilde{\lambda}_v^-]\right) + (f - [\lambda])C^+ + ([\lambda]f - e)C^- \tag{6.7}$$

Then $S(\lambda)$ is invariant under isotopy in the class of the Legendrian knots.

The proof is straightforward. One checks that $S(\lambda)$ is really invariant under all the oriented versions of non-dangerous self-tangency, triple point, cusp crossing and cusp birth moves of the wave front.

6.3.F. By taking the free homotopy classes of $\tilde{\lambda}_v^+$ and $\tilde{\lambda}_v^-$, instead of the homology classes, one obtains a different version of the $S(\lambda)$ invariant. It takes values in the group of formal half-integer linear combinations of the free homotopy classes of oriented curves in ST^*F. In this case the terms $[\lambda]$ and f in (6.7) should be substituted by the free homotopy classes of λ and a positively oriented fiber, respectively. The terms $[\lambda]f$ and e in (6.7) should be substituted by the free homotopy classes of λ, with a positive fiber added to it, and the class of a contractable curve, respectively. Note, that f lies in the center of $\pi_1(ST^*F)$, thus the class of λ, with a fiber, added is well defined.

A straightforward check shows, that this version of $S(\lambda)$ is also invariant under isotopy in the class of the Legendrian knots.

6.3.G. THEOREM. *Let L be a generic oriented cooriented wave front on an oriented surface F. Let λ be the corresponding Legendrian curve. Let $S(\lambda)$ and $l_F(\lambda)$ be the invariants introduced in* 6.3.E *and* 6.3.C, *respectively. Let*

$$\mathrm{pr} : \frac{1}{2}\mathbb{Z}\left[H_1(ST^*F)\right] \to H_1(ST^*F, \frac{1}{2}\mathbb{Z}) \tag{6.8}$$

be the mapping defined as follows: for any $n_i \in \frac{1}{2}\mathbb{Z}$ and $g_i \in H_1(ST^*F)$

$$\sum n_i g_i \mapsto \prod g_i^{n_i} \tag{6.9}$$

Then

$$\mathrm{pr}(S(\lambda)) = l_F(\lambda) \tag{6.10}$$

The proof is straightforward: one has to check that

$$[\lambda_v^+] - [\lambda_v^-] - \epsilon(v) f = [\tilde{\lambda}_v^+] - [\tilde{\lambda}_v^-] \text{ in } H_1(ST^*F). \tag{6.11}$$

(Here we use an additive notation for the group operation in $H_1(ST^*F)$.)

This means that $S_F(\lambda)$ is a splitting of $l_F(\lambda)$.

6.3.H. One can check, that there is a unique linear combination $(\sum_{m\in\mathbb{Z}} n_m[m]_q)$ (with n_m, being non-negative half-integers, such that, $n_0 = 0$ and, if $n_m > 0$, then $n_{-m} = 0$) which corresponds to $l_q(\lambda) \in \frac{1}{2}\mathbb{Z}[q, q^{-1}]$. To prove this, one checks, that $\{\frac{1}{2}[n]_q | 0 < n\}$ is a basis for the $\mathbb{Z}$-submodule of $\frac{1}{2}\mathbb{Z}[q, q^{-1}]$, generated by the quantum numbers, and uses the identity $n[m]_q = -n[-m]_q$.

The following theorem shows, that, if $L \subset \mathbb{R}^2$, then $l_q(\lambda)$ (see 6.3.B) and $S(\lambda)$ can be explicitly expressed through each other.

6.3.I. THEOREM. *Let $f \in H_1(ST^*\mathbb{R}^2)$ be the class of a positively oriented fiber. Let L be a generic oriented cooriented wave front on $\mathbb{R}^2$, λ be the corresponding Legendrian curve, and f^h be the homology class, represented by it.*

Let $\phi : \frac{1}{2}\mathbb{Z}[H_1(ST^*\mathbb{R}^2)] \to \frac{1}{2}\mathbb{Z}[q, q^{-1}]$ be the mapping defined as follows: for any $n \in \frac{1}{2}\mathbb{Z}$ $m \in \mathbb{Z}$

$$\phi(nf^m) = \begin{cases} n[2m-h-1]_q, & \text{if } n > 0 \\ 0, & \text{otherwise.} \end{cases} \tag{6.12}$$

For odd (even) $h \in \mathbb{Z}$ let ψ be the mapping from the set of half-integer linear combinations of even (odd) quantum numbers to $\frac{1}{2}\mathbb{Z}[H_1(ST^*\mathbb{R}^2)]$ defined as follows: for any $n \in \frac{1}{2}\mathbb{Z}$ and any even *(odd)* $m \in \mathbb{Z}$

$$\psi(n[m]_q) = n(f^{\frac{m+h+1}{2}} - f^{\frac{h+1-m}{2}}) \tag{6.13}$$

Then

$$l_q(\lambda) = \phi(S(\lambda)) + [h]_q h \tag{6.14}$$

For $l_q(\lambda)$ *written in the form, described in* 6.3.H

$$S(\lambda) = \psi(l_q(\lambda) - [h]_q h) \tag{6.15}$$

(One can check that ψ is really defined for $l_q(\lambda) - [h]_q h$.) For the proof of Theorem 6.3.I see Section 8.11.

Note, that the $l_q(\lambda)$ invariant was defined only for fronts on the $\mathbb{R}^2$ plane. Thus, $S(\lambda)$ is, in a sense, a generalization of $l_q(\lambda)$ to the case of wave fronts on an arbitrary oriented surface F.

6.3.J. The splitting of the knot K into $\mu_1^+(v)$, $\mu_1^-(v)$, $\mu_2^+(v)$ and $\mu_2^-(v)$, used to introduce $\tilde{S}(K)$ (see 4.4.A), can be done up to an isotopy in the class of the Legendrian knots. Although this can be done in many ways, there is the simplest one. The projections $\tilde{L}_1^+(v)$, $\tilde{L}_1^-(v)$, $\tilde{L}_2^+(v)$ and $\tilde{L}_2^-(v)$ are shown in Figure 16. (This fact follows from the Theorem 6.2.C).

This allows us to introduce an invariant similar to $S(\lambda)$ for generic oriented cooriented wave fronts on a non-orientable surface F in the following way.

Let L be a generic wave front on the non-orientable surface F. Let v be a crossing point of L. Fix some orientation in the small neighborhood of v in F. Then this neighborhood can be identified with a model picture, shown in Figure 16. We split our wave front at this point with respect to orientation and coorientation as it is shown in Figure 16. We correspond to each crossing point v of L an element $\left([\tilde{\lambda}_1^+(v)] - [\tilde{\lambda}_1^-(v)] + [\tilde{\lambda}_2^+(v)] - [\tilde{\lambda}_2^-(v)]\right) \in \mathbb{Z}[H_1(ST^*(F)]$. (Here lambdas are the corresponding Legendrian curves). Clearly, this element does not depend on the orientation we picked in the neighborhood of v.

For a wave front L let C be half of the number of cusps of L. Denote by f the homology class of the fiber of ST^*F oriented in some way. Note, that $f^2 = e$, so it does not matter, which orientation of the fiber we use to define f.

6.3.K. THEOREM. *Let L be a generic cooriented oriented wave front on a non-orientable surface F and λ be the corresponding Legendrian curve. Define $\tilde{S}(\lambda) \in \frac{1}{2}\mathbb{Z}\left[H_1(ST^*(F))\right]$ by the following*

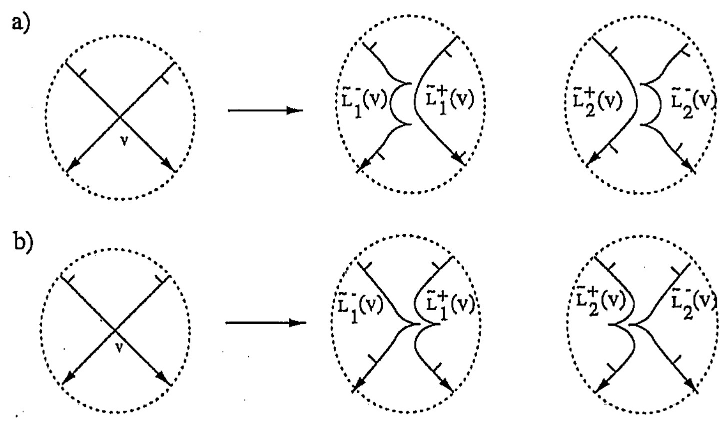

FIGURE 16. Nonsymmetric splitting of a wave front on a nonorientable surface.

formula, where the summation is taken over the set of all the double points of L.

$$\tilde{S}(\lambda) = \sum_v \Big([\tilde{\lambda}_1^+(v)] - [\tilde{\lambda}_1^-(v)] + [\tilde{\lambda}_2^+(v)] - [\tilde{\lambda}_2^-(v)]\Big) + C([\lambda]f - e + f - [\lambda]) \tag{6.16}$$

Then $\tilde{S}(\lambda)$ is invariant under isotopy in the class of the Legendrian knots.

The proof is straightforward. One checks that $\tilde{S}(\lambda)$ is really invariant under all the oriented versions of non-dangerous self-tangency, triple point passing, cusp crossing and cusp birth moves of the wave front.

The reason we have $\tilde{S}(\lambda) \in \frac{1}{2}\mathbb{Z}[H_1(ST^*F)]$ is, that if L is an orientation reversing curve, then the number of cusps of L is odd. In this case C is a half-integer.

Similarly to 6.3.F, one can introduce the version of the $\tilde{S}(\lambda)$ invariant, which takes values in the group of formal half-integer linear combinations of all the free homotopy classes of oriented curves in ST^*F.

7. WAVE FRONTS ON ORBIFOLDS

7.1. Definitions.

7.1.A. DEFINITION. An *orbifold* is a surface F with the additional structure, which consists of:

1) A set $A \subset F$.

2) A smooth structure on $F \setminus A$.

3) A set of homeomorphisms ϕ_a of a neighborhood U_a of a in F onto $\mathbb{R}^2/G_a$, such that $\phi_a(a) = 0$ and $\phi_a\big|_{U_a \setminus a}$ is a diffeomorphism. Here $G_a = \{e^{\frac{2\pi k}{\mu_a}} \,|\, k \in \{1, \dots, \mu_a\}\}$ ($\mu_a > 0$) is a group, which acts

on $\mathbb{R}^2 = \mathbb{C}$ by multiplication.

The points $a \in A$ are called *cone points.*

The action of G on $\mathbb{R}^2$, induces the action of G on $ST^*\mathbb{R}^2$. This turns $ST^*\mathbb{R}^2/G$ into a Seifert fibration over $\mathbb{R}^2/G$. Gluing together the pieces over neighborhoods of F, we obtain a Seifert fibration $\pi : N \to F$. The fiber over a cone point a is an exceptional fiber of type $(\mu_a, -1)$ (see 5.0.E).

The space $ST^*\mathbb{R}^2$ has a natural contact structure, which is invariant under the induced action of G. Since G acts freely on $ST^*\mathbb{R}^2$, this implies, that N has an induced contact structure. As before, we call the projection $L \subset F$ of a cooriented Legendrian curve λ *the front of* λ.

7.2. Invariants of fronts on orbifolds. If F is oriented, we construct an invariant similar to $S(\lambda)$. It corresponds to the S_K invariant of a knot in a Seifert fibered space. If F is a non-orientable surface, then we construct an analogue of $\tilde{S}(\lambda)$. It corresponds to the $\tilde{S}_K$ invariant of a knot in a Seifert fibered space.

Note, that any surface F can be viewed as an orbifold without any cone points. This justifies the notation below.

Let F be an oriented surface. The orientation of F induces an orientation on all the fibers. Denote by f the homology class of a positively oriented fiber. For a cone point a denote by f_a the homology class of a positively oriented fiber $\pi^{-1}(a)$. For a generic oriented cooriented wave front $L \subset F$ denote by C^+ (C^-) half of the number of positive (negative) cusps of L. Note, that for a double point v of a generic front L, the splitting into $\tilde{L}_v^+$ and $\tilde{L}_v^-$ is well defined. The corresponding Legendrian curves $\tilde{\lambda}_v^+$ and $\tilde{\lambda}_v^-$ in N are also well defined.

For $a \in A$ of type $(\mu_a, -1)$ put $N_1(a) = \{k \in \{1, \dots, \mu_a\} | \frac{-2k\pi}{\mu_a} \bmod 2\pi \in (0, \pi]\}$ $N_2(a) = \{k \in \{1, \dots, \mu_a\} | \frac{2k\pi}{\mu_a} \bmod 2\pi \in (0, \pi)\}$ Define $R_a^1, R_a^2 \in \mathbb{Z}[H_1(N)]$ by the following formulas

$$R_a^1 = \sum_{k \in N_1(a)} ([K] f_a^{\mu_a - k} - f_a^k) - \sum_{k \in N_2(a)} (f_a^{\mu_a - k} - [K] f_a^k) \tag{7.1}$$

$$R_a^2 = \sum_{k \in N_1(a)} (f_a^{\mu_a - k} - [K] f_a^k) - \sum_{k \in N_2(a)} ([K] f_a^{\mu_a - k} - f_a^k). \tag{7.2}$$

Set J to be $\frac{1}{2}\mathbb{Z}[H_1(N)]$, factorized by the Abelian subgroup generated by $\left\{ \{\frac{1}{2}R_1(a), \frac{1}{2}R_2(a)\}_{a \in A} \right\}$.

7.2.A. THEOREM. *Let L be a generic cooriented oriented wave front on F and λ be the corresponding Legendrian curve.*

Then $S(\lambda) \in J$ defined by the sum over all the double points of L

$$S(\lambda) = \sum\Big([\tilde{\lambda}^{+}(v)] - [\tilde{\lambda}^{-}(v)]\Big) + (f - [\lambda])C^{+} + ([\lambda]f - e)C^{-} \tag{7.3}$$

is invariant under isotopy in the class of the Legendrian knots.

For the proof of Theorem 7.2.A see Section 8.12.

Let F be a non-orientable surface. Denote by f the homology class of a regular fiber oriented in some way. Note, that $f^2 = e$, so the orientation, we use to define, f does not matter. For a cone point a denote by f_a the homology class of the fiber $\pi^{-1}(a)$, oriented in some way. For a generic oriented cooriented wave front $L \subset F$ denote by C half of the number of cusps of L. Note, that for a double point v of a generic front L, the element $([\tilde{\lambda}_1^{+}(v)] - [\tilde{\lambda}_1^{-}(v)] + [\tilde{\lambda}_2^{+}(v)] - [\tilde{\lambda}_2^{-}(v)]) \in \mathbb{Z}[H_1(N)]$ used to introduce $\tilde{S}(\lambda)$ is well defined.

For $a \in A$ of type $(\mu_a, -1)$ put $N_1(a) = \big\{k \in \{1, \dots, \mu_a\} \big| \frac{-2k\pi}{\mu_a} \bmod 2\pi \in (0, \pi]\big\}$ and $N_2(a) = \big\{k \in \{1, \dots, \mu_a\} \big| \frac{-2k\pi}{\mu_a} \bmod 2\pi \in (0, \pi)\big\}$ Define $R_a \in \mathbb{Z}[H_1(N)]$ by the following formula

$$R_a = \sum_{k \in N_1(a)} \Big([\lambda]f_a^{\mu_a - k} - f_a^{k} + f_a^{k-\mu_a} - [\lambda]f_a^{-k}\Big) - \sum_{k \in N_2(a)} \Big(f_a^{\mu_a - k} - [\lambda]f_a^{k} + [\lambda]f_a^{k-\mu_a} - f_a^{-k}\Big) \tag{7.4}$$

Put $\tilde{J}$ to be $\frac{1}{2}\mathbb{Z}[H_1(N)]$, factorized by a free Abelian subgroup generated by $\Big\{\{\frac{1}{2}R_a\}_{a \in A}\Big\}$.

Similarly to 5.0.H one can prove, that under the change of the orientation of $\pi^{-1}(a)$ (used to define f_a), R_a goes to $-R_a$. Thus, $\tilde{J}$ is well defined.

7.2.B. THEOREM. *Let L be a generic cooriented oriented wave front on F and λ be the corresponding Legendrian curve.*

Then $\tilde{S}(\lambda) \in \tilde{J}$ defined by the summation over all the double points of L

$$\tilde{S}(\lambda) = \sum\Big([\tilde{\lambda}_1^{+}(v)] - [\tilde{\lambda}_1^{-}(v)] + [\tilde{\lambda}_2^{+}(v)] - [\lambda_2^{-}(v)]\Big) + (([\tilde{\lambda}]f - e + f - [\lambda])C \tag{7.5}$$

is invariant under isotopy in the class of the Legendrian knots.

The proof is a straightforward generalization of the proof of Theorem 7.2.A.

As before (see 6.3.F and 5.0.J), one can introduce versions of $S(\lambda)$ and $\tilde{S}(\lambda)$, taking values in the factors of the group of all formal half-integer linear combinations of the free homotopy classes of oriented curves in N.

8. Proofs

8.1. Proof of Theorem 2.3.E. I: K' can be obtained from K by a sequence of isotopies and modifications along fibers. Isotopies do not change $\tilde{U}$. The modifications change $\tilde{U}$ by elements of $2G_K$. Thus, the first part of the theorem is proved.

II: We prove that for any $g \in G_K$ there exist two knots K_1 and K_2 such, that they represent the same free homotopy class as K and

$$\tilde{U}_{K_1} = \tilde{U}_K - 2g \tag{8.1}$$

$$\tilde{U}_{K_2} = \tilde{U}_K + 2g \tag{8.2}$$

Clearly, this implies the second statement of the theorem. To obtain the two knots we isotopically deform K so that $\pi(K)$ bites itself in the projection (as it is shown in Figure 17) and $G_u = G_v = g$. To obtain K_1, one performs a fiber modification along $\pi^{-1}(u)$. To obtain K_2, one performs a fiber modification along $\pi^{-1}(v)$.

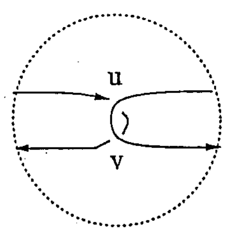

Figure 17. The knot bites itself.

This finishes the proof of Theorem 2.3.E. □

8.2. Proof of Theorem 2.4.E. Let D be a meridional disc along the boundary of which, we performed the positive Dehn twist (used to define Φ). Assume, that all the branches of K, which cross D, are perpendicular to it and are located on different levels (see Figure 18). Using second Reidemeister moves transform the diagram in such a way, that if we traverse K along the orientation, then the branches cross D in the order shown in Figure 18. (The thick dashed line in Figure 18 is $p(D)$). After we compose the embedding of T with Φ, the diagram will be changed, as it is shown in Figure 19.

Note, that $A(K)$ and $A'(K)$ change in the same way under the modification of pushing of one branch of the knot through the other, which happens outside of the neighborhood of D (shown in

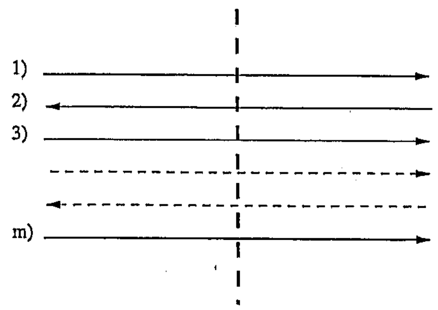

FIGURE 18. Branches of the knot before the automorphism.

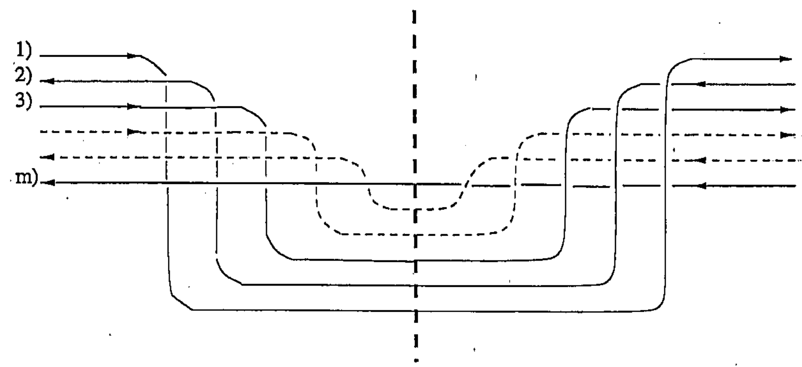

FIGURE 19. Branches of the knot after the automorphism.

Figure 19). Hence, their difference is preserved. Thus, we can assume that our knot K has an ascending diagram. After a simple calculation we get the desired result. □

8.3. Proof of Theorem 2.4.G. The relation between U and A invariants, shown in 2.4.B, allows one to use 2.3.F in the case of a partial linking polynomial. There is a natural bijection between one-dimensional homology classes of T and free homotopy classes of oriented loops in T.

Thus we get, that:

a) If K and K', are such that $[p(K)] = [p(K')] = h$, then $A(K')$ and $A(K)$ are congruent modulo the additive subgroup generated by all the elements of type

$$\pm (t^j + t^{h-j}) \text{ for } j \notin \{h, 0\} \tag{8.3}$$

(Note that if $h = 2j$ then this expression is equal to $\pm 2t^j$.)

b) Let K be a knot (with $[p(K)] = h$), and let A be a finite Laurent polynomial congruent to $A(K)$ modulo the additive subgroup, generated by all the elements of type (8.3). Then there exists a knot K', such that $[p(K')] = h$ and $A(K') = A$.

Thus, if K and K' are knots such, that $[p(K)] = [p(K')] = h$ and $A(K) \in P_h$, then $A(K') \in P_h$. And vice versa, if for some $p_h \in P_h$ there exists a knot K_{p_h}, such that $[p(K_{p_h})] = h$ and $A(K_{p_h}) = p_h$, then it exists for any $\tilde{p}_h \in P_h$. Hence, to prove the theorem it is sufficient to show, that for any $h \in \mathbb{Z}$ there exists a knot K_h, such that $[p(K_h)] = h$ and $A(K_h) \in P_h$. Let K_h be a knot, that rotates h times in T and has an ascending diagram (see Figure 20). The A invariant of it is equal to (8.4) and it belongs to P_h.

$$\begin{cases} t^1 + t^2 + \cdots + t^{h-1}, & \text{if } h > 0 \\ t^{-1} + t^{-2} + \cdots + t^{h+1}, & \text{if } h < 0 \\ 0, & \text{if } h = 0 \end{cases} \tag{8.4}$$

This finishes the proof of Theorem 2.4.G. □

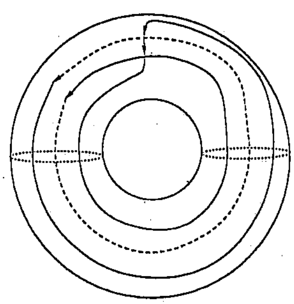

FIGURE 20. Knot with an ascending diagram.

8.4. Proof of Theorem 4.1.C. To prove the theorem, it is sufficient to show, that S_K is invariant under the elementary isotopies. They project to death of a double point, cancelation of two double points and passing through a triple point.

To prove the invariance, we fix a part P of our surface, such that P is homeomorphic to a closed disc, in which one of these events takes place. Fix a section over the boundary of this disc, such that the points of $K \cap \pi^{-1}(\partial P)$ belong to the section. Inside P we can construct the Turaev shadow (see 3.2.K). The action of $H_1(\text{Int } P) = e$ on the set of isotopy types of links is trivial (see 3.2.H). Thus, the part of K can be reconstructed in the unique way from the shadow over P. In particular one can compare the homology classes of the curves created by splitting at a crossing point inside P. Hence, to prove the theorem it is sufficient to check the invariance under the oriented versions of S_1, S_2 and S_3 moves.

There are two versions of the oriented move S_1, shown in Figure 21a and Figure 21b.

The $[\mu_v^+]$ for the Figure 21a appears to be equal to f. From 4.1.B we know, that $[\mu_v^+][\mu_v^-] = [K]f$, thus $[\mu_v^-] = [K]$. Hence, $[\mu_v^+] - [\mu_v^-] = f - [K]$ and is equal to zero in H. In the same way we check, that $[\mu_v^+] - [\mu_v^-]$, for v shown in Figure 21b, is equal to $[K]f - e$. It is also zero in H. The summands, appearing from the other vertices, are not changed under this move, as it does not change homology classes of the knots, created by these splittings.

There are three oriented versions of the S_2 move. We show, that S_K is not changed under one of them. The proof for the other two is the same or easier. We choose the version, in which participates the upper part of Figure 22.

The summands, appearing from the vertices not in this figure, are preserved under the move, as it does not change the homology classes of the corresponding knots. So it is sufficient to show, that the terms, produced by the vertices u and v in this figure cancel out. Note, that the shadow μ_v^- is transformed to μ_u^+, by the $\bar{S}_1$ move. It is known, that $\bar{S}_1$ can be expressed through S_1, S_2 and S_3, thus, it also does not change the homology classes of the knots created by the splittings. Hence, $[\mu_u^+]$ and $[\mu_v^-]$ cancel out. In the same way one proves, that $[\mu_u^-]$ and $[\mu_v^+]$ also cancel out.

There are two oriented versions of the S_3 move: S_3' and S_3'', shown in Figure 23a and Figure 23b, respectively. The S_3'' move can be expressed through S_3' and oriented versions of S_2 and S_2^{-1}. To prove this, we use Figure 24. There are two ways to get from Figure 24a to Figure 24b. One is to apply S_3''. Another way is to apply three times oriented versions of S_2, to obtain Figure 24c, then S_3', to get Figure 24d, and finally use three times the oriented versions of S_2^{-1}, to finish at Figure 24b.

Thus, it is sufficient to check the invariance under S_3'. The terms, appearing from the vertices not in Figures 25a and 25b are preserved, because of the same reasons as above. It happens, that the terms coming from vertices u in Figure 25a and u in Figure 25.b are the same. This holds also for the v- and w- pairs of vertices on these two figures. We prove this statement only for the u-pair of vertices. For v- and w- pairs the proof is the same or simpler. There is only one possibility: either the dashed line belongs to both $\pi(\mu_u^+)$ in Figure 25a.1 and Figure 25b.1, respectively, or to both $\pi(\mu_u^-)$ in Figure 25a.2 and Figure 25b.2, respectively. We choose the one, it does not belong to. Summing gleams on each of the two sides of it, we immediately see, that the corresponding shadows are the same on both pictures. Thus, the homology classes of the corresponding knots are equal. But $[\mu_u^+][\mu_u^-] = [K]f$, (see 4.1.B) thus, the homology classes of the knots, represented by the other shadows are also equal, and we are done. This finishes the proof of Theorem 4.1.C.

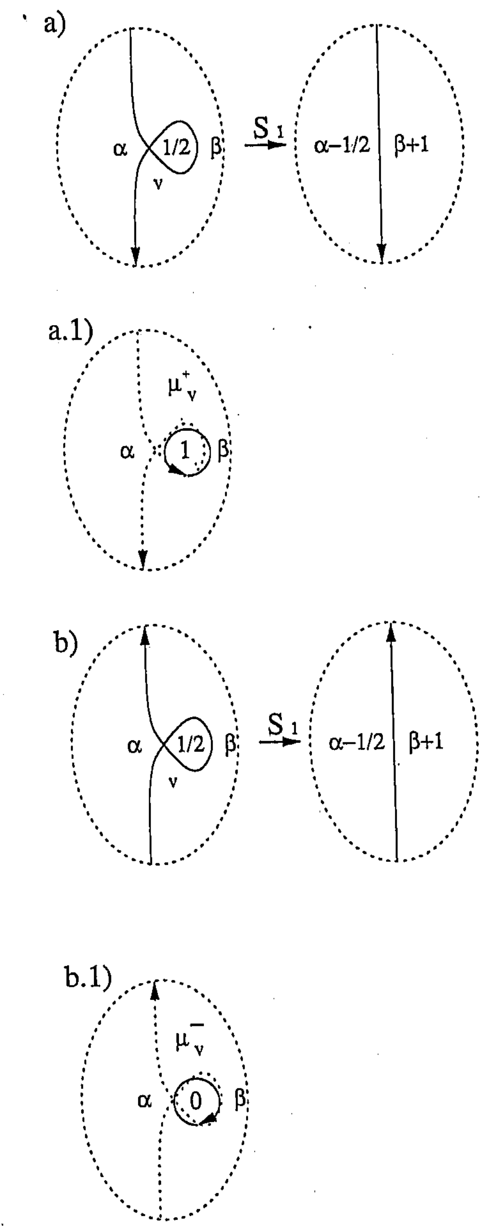

FIGURE 21. Invariance of S_K under the first shadow move.

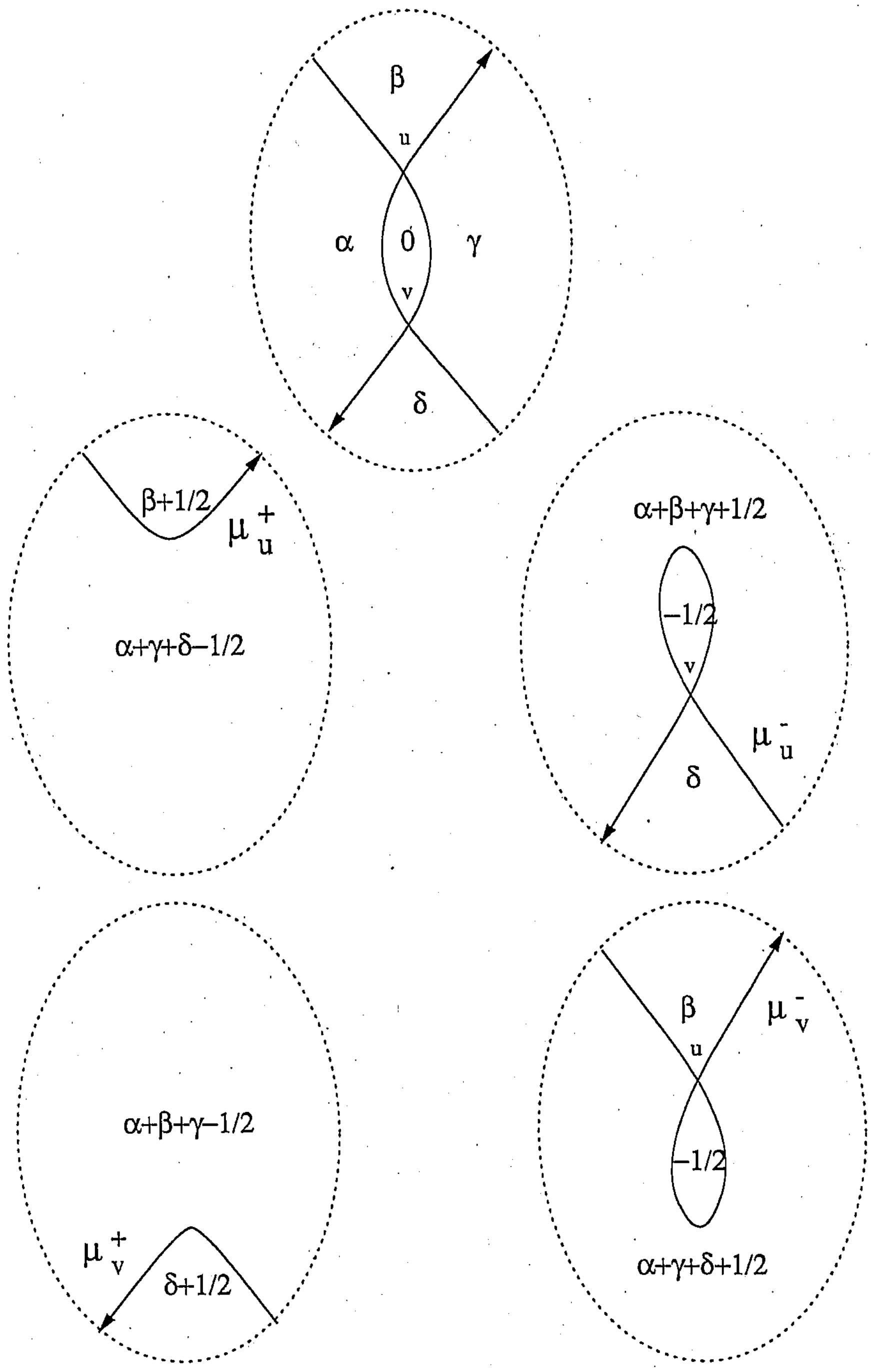

FIGURE 22. Invariance of S_K under the second shadow move.

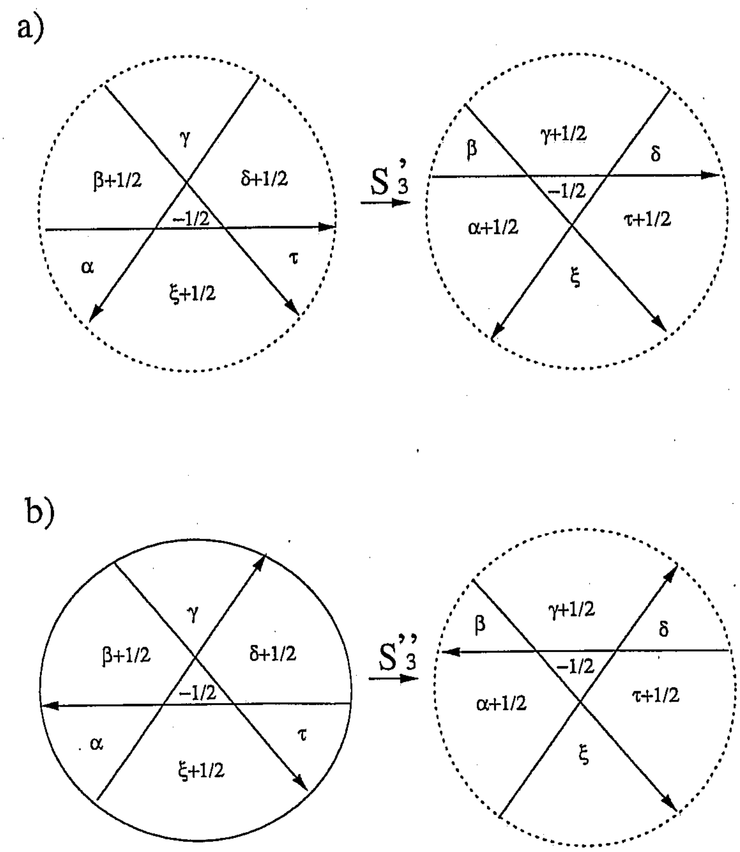

FIGURE 23. Two oriented versions of the third shadow move.

8.5. Proof of Theorem 4.2.C. I: K' can be obtained from K by a sequence of isotopies and modifications along fibers. Isotopies do not change S. The modifications change S by elements of type (4.3). Finally to finish the proof we use the identity $\xi_1\xi_2 = [K]$.

II: We prove, that for any $i \in H_1(N)$ there exist two knots K_1 and K_2, such that they represent the same free homotopy class as K and

$$S_{K_1} = S_K + (f-e)([K]i^{-1} + i) \tag{8.5}$$

$$S_{K_2} = S_K - (f-e)([K]i^{-1} + i) \tag{8.6}$$

Clearly, this would imply the second statement of the theorem.

Take $i \in H_1(N)$. Let K_i be an oriented knot in N, such that $[K_i] = i$. The space N is oriented, hence the tubular neighborhood T_{K_i} of K_i is homeomorphic to an oriented solid torus T. Deform K_i, so that $K_i \cap T_{K_i}$ is a small arc (see Figure 26). Pull one part of the arc along K_i in T_{K_i} under

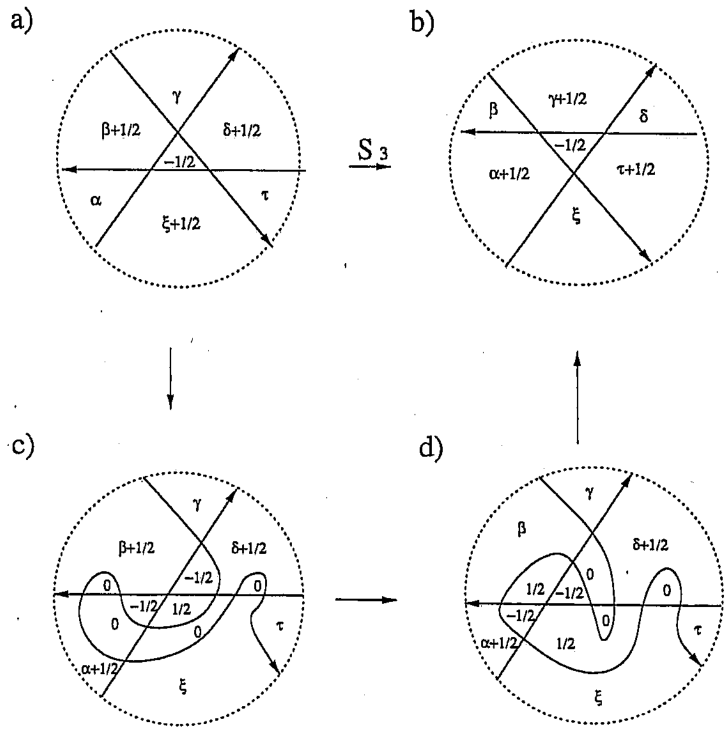

FIGURE 24. Expression of S_3'' through the other moves.

the other part of the arc, as shown in Figure 27. This isotopy creates two new vertices u and v of $\pi(K)$. (As T_{K_i} may be knotted, it can be that there are other new vertices, but we do not need them for our construction.) By making a fiber modification along the part of $\pi^{-1}(u)$ that lies in T, one obtains K_2. By making a fiber modification along the part of $\pi^{-1}(v)$ that lies in T, one obtains K_1. This finishes the proof of Theorem 4.2.C. □

8.6. Proof of Theorem 4.3.C. It is easy to check, that any two shadows, having the same projection, can be transformed to each other by a sequence of fiber fusions. One can easily create a trivial knot, having an ascending diagram, such that its projection is any desired curve. This implies, that any two shadows on $\mathbb{R}^2$ can be transformed to each other by a sequence of fiber fusions and movements S_1, S_2, S_3 and their inverses. A straightforward check shows, that $\sigma(s(K))$ is not changed under the S_1, S_2, S_3 moves and their inverses. Under fiber fusions the homology class of the knot and σ are changed in the same way. To prove this, we use Figure 28, where Figure 28a shows the shadow before the application of the fiber fusion (that adds 1 to the homology class of the knot) and Figure 28b after. In this diagrams the indices of the regions are denoted by Latin

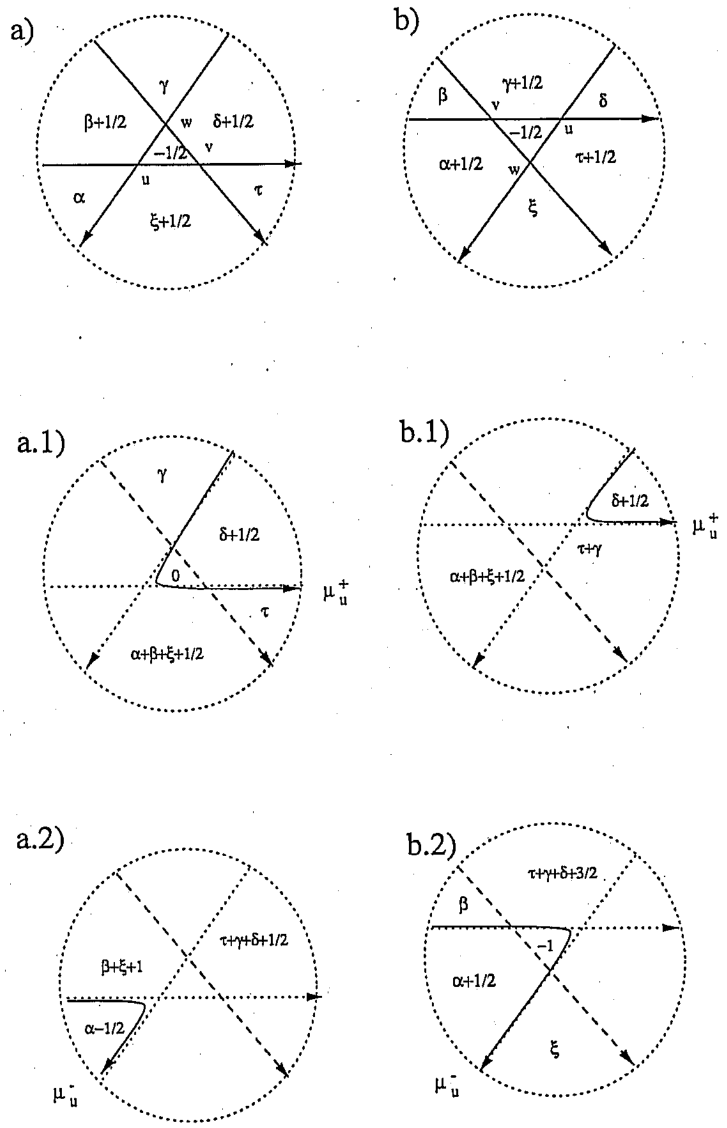

FIGURE 25. Invariance of S_K under the third shadow move.

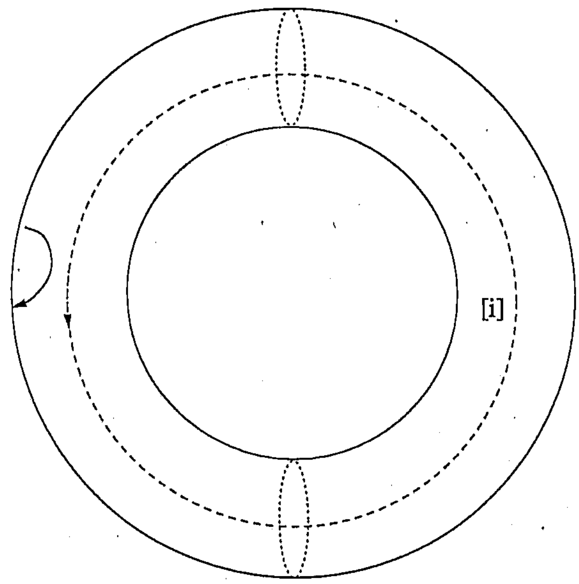

FIGURE 26. Desired deformation of a knot.

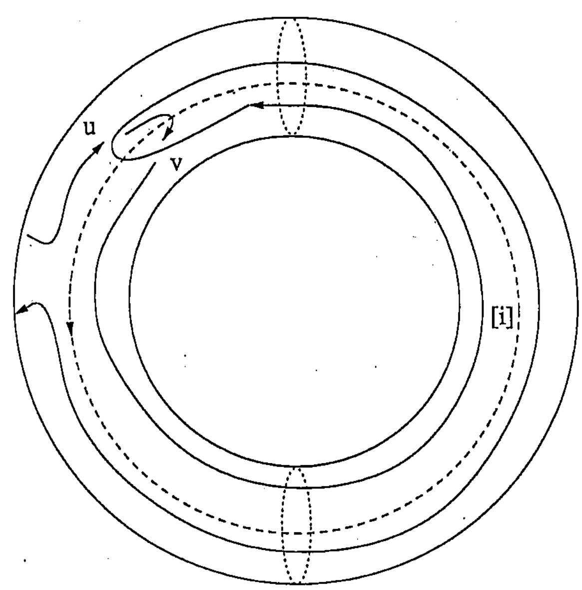

FIGURE 27. Pulling of one branch of the knot under the other.

letters. Now one easily verifies, that σ is also changed by one. Finally, for the trivial knot with a trivial shadow diagram the homology class of it and $\sigma(s(K))$ are both equal to 0. This finishes the

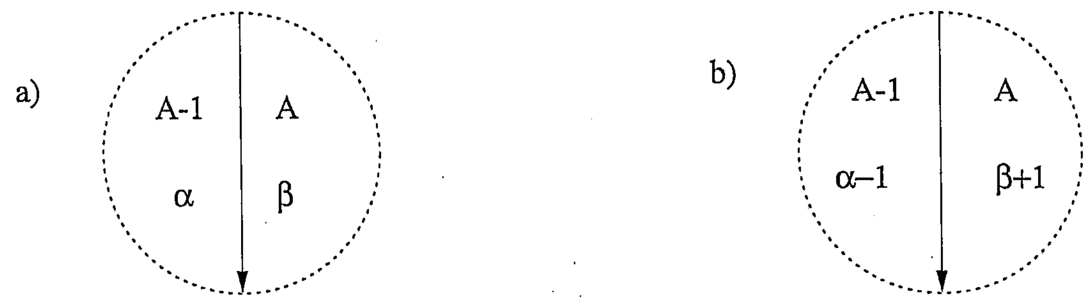

FIGURE 28. Change of the shadow under the fiber fusion.

proof of Theorem 4.3.C. □

8.7. Proof of Theorem 4.2.E. The knot K can be transformed to $\tilde{K}$ inside our $\mathbb{R}^1$-fibration E by a sequence of isotopies and modifications along fibers. We compare the variations of S and U under the procedure, described above. Let v be a crossing point of the diagram, appearing after the modification and v' be the corresponding crossing point before the modification. Assume that $\omega_v = 1$, as in Figure 29a. Then before the surgery the diagram was as in Figure 29b. Each of the two crossing points contributes $\pm\frac{1}{2}$ to the gleams of the adjacent regions, as it is shown in Figure 29a and Figure 29b. From Figures 29 a, b, c, and d, one obtains that

$$[\mu_v^+] = \xi_2(v)f \quad [\mu_v^-] = \xi_1(v) \quad [\mu_{v'}^+] = \xi_2(v') \quad [\mu_{v'}^-] = \xi_1(v')f \tag{8.7}$$

Note that $\xi_1(v) = \xi_1(v')$ and $\xi_2(v) = \xi_2(v')$. Thus, the change $\Delta S_v'$ (see 4.1.D) for the surgery, appears to be equal to

$$
\Delta S_v = \begin{cases}
\xi_2(v)f - \xi_1(v) + \xi_1(v)f - \xi_2(v) \\
= (f-e)(\xi_1(v)+\xi_2(v)), \\
\quad \text{if } \left\{e, f\right\} \cap \left\{\xi_1(v), \xi_2(v), \xi_1(v)f, \xi_2(v)f\right\} = \varnothing, \\
\xi_2(v)f - \xi_1(v), \\
\quad \text{if } \left\{e, f\right\} \cap \left\{\xi_2(v)f, \xi_1(v)\right\} = \varnothing \text{ and } \left\{e, f\right\} \cap \left\{\xi_2(v), \xi_1(v)f\right\} \neq \varnothing, \\
\xi_1(v)f - \xi_2(v), \\
\quad \text{if } \left\{e, f\right\} \cap \left\{\xi_2(v), \xi_1(v)f\right\} = \varnothing \text{ and } \left\{e, f\right\} \cap \left\{\xi_2(v)f, \xi_1(v)\right\} \neq \varnothing \\
0, \text{ otherwise.}
\end{cases} \tag{8.8}
$$

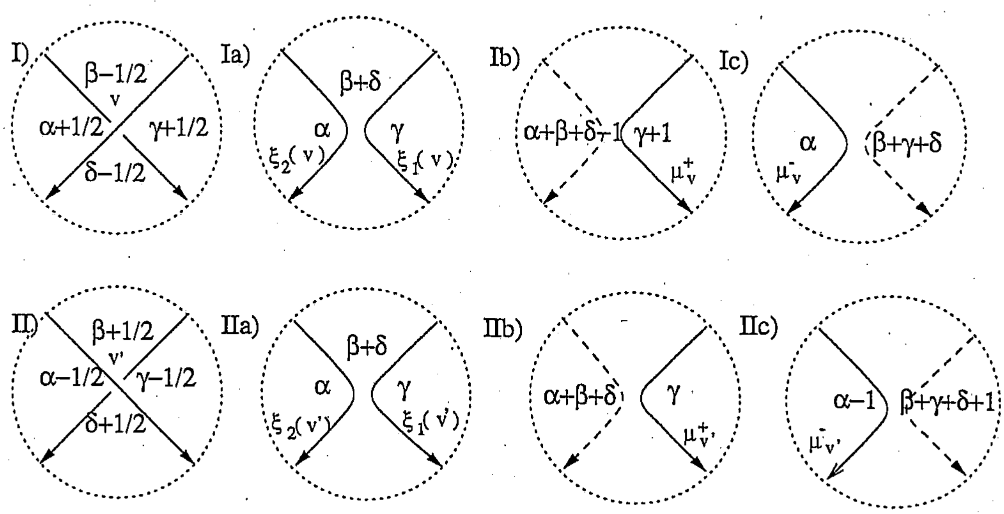

FIGURE 29. Contribution of the crossing point of the knot diagram to its shadow.

The change ΔU_v is given by the following formula

$$
\Delta U_v = \begin{cases}
2(\xi_1(v)+\xi_2(v)), & \text{if } e \notin \{\xi_1(v), \xi_2(v)\} \\
0, & \text{otherwise.}
\end{cases} \tag{8.9}
$$

For a homology class i denote by α_i the coefficient of i in ΔU_K. One checks that $\frac{1}{2}\alpha_i$, is equal to the sum of the signs of the vertices created by the modifications (at the vertices, where the homology class of one of the two curves, created by splitting is i), that have to be made to transform K to $\tilde{K}$.

Formulas (8.8) allow one to calculate the change of S' produced by these modifications. After some calculations one obtains the desired result. This finishes the proof of Theorem 4.2.E. □

8.8. Proof of Theorem 4.3.F. Denote the $\mathbb{R}^1$- and S^1- fibration we used to define A and S' by p and π, respectively. Deform the knot K in such a way that the set Q of the preimages of all the crossing points of $p(K)$ is contained in a small cylinder Z, whose axis is parallel to the kernel of p. Deform the fiber structure of π in such a way, that inside Z the fibers of π are parallel to the generator element of Z (see Figure 30). This fiber structure is ambient isotopic to the original, hence

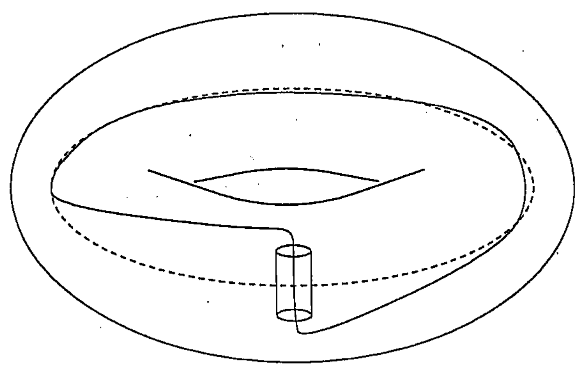

FIGURE 30. Deformation of the fiber structure.

this deformation does not change S'_K. If one looks along the new fibers he sees all the crossings from Q (plus some new crossings, but we are not interested in them). Using modifications along the $\mathbb{R}^1$ fibers over Q one can transform K to a knot $\tilde{K}$, that has an ascending diagram with respect to the $\mathbb{R}^1$- fiber structure. This $\tilde{K}$ isotopic to the knot, shown in Figure 20. The $A(\tilde{K})$ invariant for this knot is equal to A^h. On the other hand, the shadow for $\tilde{K}$ can be transformed by a sequence of the S_1, S_2 and S_3 moves and their inverses to the shadow shown in Figure 31a. This diagram does

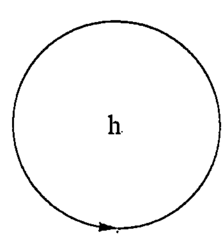

FIGURE 31. Trivial shadow.

not have any crossing points thus the S' invariant of the corresponding knot is 0. But this knot is isotopic to $\tilde{K}$. Thus $S'(\tilde{K}) = 0$.

Finally one repeats the corresponding step of the proof of 4.2.E and gets that

$$S'_K = (A(K) - A^h)(t-1) + N_h \tag{8.10}$$

where

$$N_h = \begin{cases} -\frac{1}{2}\alpha_{-1}(1-t^{h+1}) - \frac{1}{2}\alpha_1(t^h - t), & \text{if } h \notin \{0, \pm 1, \pm 2\} \\ -\frac{1}{2}\alpha_1(1-t), & \text{if } h = 0 \\ -\frac{1}{2}\alpha_{-1}(t-1)t^{-1} & \text{if } h = -2 \\ -\frac{1}{2}\alpha_1(t-1)t & \text{if } h = 2 \\ 0, & \text{otherwise} \end{cases} \tag{8.11}$$

To finish the proof one has to split out the $t-1$ term from N_h. □

8.9. Proof of Theorem 5.0.G. It is enough to show, that S_K, defined in this way does not change under elementary isotopies of the knot. Three of them correspond in the projection to a birth of a small loop, passing through a point of self tangency and passing through a triple point. The fourth one is passing through an exceptional fiber.

From the proof of Theorem 4.1.C one gets, that S_K is invariant under the first three of the elementary isotopies, described above. Thus, it is sufficient to prove invariance under the move of passing through an exceptional fiber a.

Let a be a singular fiber of type (μ_a, ν_a) (see 5.0.E). Let T_a be its neighborhood, fiber-wise isomorphic to the standardly fibered solid torus with an exceptional fiber of type (μ_a, ν_a).

We can assume, that the move happens as follows. At the start K and T_a intersect along a curve, lying in the meridional disc D of T_a. The part of K close to a in D is an arc C of a circle of a very big radius. This arc is symmetric through the y axis passing through a in D. During the move this arc slides along the y axis through the fiber a (see Figure 32).

Clearly, two points u and v of C after this move are in the same fiber iff they are symmetric with respect to the y axis and the angle formed by v, a, u in D is less or equal to π and is equal to $\frac{2l\pi}{\mu_a}$ for some $l \in \{1, \dots, \mu_a\}$ (see Figure 32). They are in the same fiber before the move if the angle formed by u, a, v in D is less than π and is equal to $\frac{2l\pi}{\mu_a}$ for some $l \in \{1, \dots, \mu_a\}$ (see Figure 32).

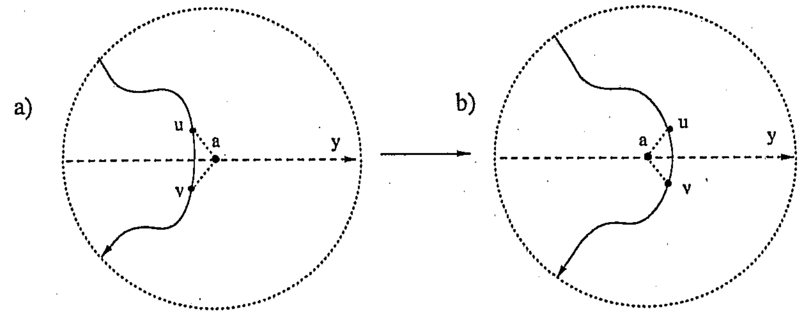

FIGURE 32. Branch of the knot passes through the singular fiber.

Consider a double point v of $\pi\big|_D(K)$ that appears after the move and corresponds to the angle $\frac{2l\pi}{\mu_a}$. There is a unique $k \in N_1(a)$, such that $\frac{2\pi\nu_a k}{\mu_a} \bmod 2\pi = \frac{2l\pi}{\mu}$. Note that for the splitting of $[K]$ into $[\mu_v^+]$ and $[\mu_v^-]$ to be well defined, we don't need the two preimages of v to be antipodal in $\pi^{-1}(v)$. This allows one to compare these homology classes to f_a. For the orientation of C shown in Figure 32 one checks that connecting v to u along the orientation of the fiber we are adding in fact k fibers f_a. (Note that the factorization we used to define the exceptionally fibered torus was $((r,\theta),1) = ((r,\theta+\frac{2\pi\nu}{\mu}),0)$.) Thus $[\mu_v^-] = f_a^k$ (see Figure 33). From 4.1.B we know that $[\mu_v^+][\mu_v^-] = [K]f$. Hence $[\mu_v^+] = [K]f_a^{\mu_a-k}$.

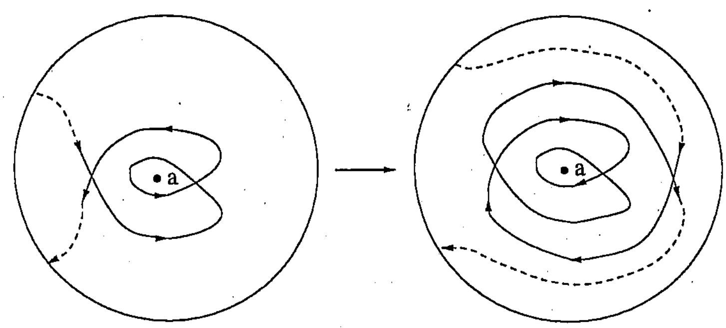

FIGURE 33. Projection of the knot branch passing through a singular fiber.

As above to each double point v of $\pi\big|_D(K)$ before this move corresponds $k \in N_2(a)$. For this double point $[\mu_v^+] = f_a^{\mu_a-k}$ and $[\mu_v^-] = [K]f_a^k$.

Making sums over the corresponding values of k, we get, that the value of the jump of S_K under this move is R_a^1. Hence it is zero in H.

For the other orientation of C we get the value of the jump equal to R_a^2.

Thus S_K is invariant under all the elementary isotopies, and this proves the theorem. □

8.10. Proof of Theorem 6.2.C. In the case, when F is oriented the proof is as follows. Deform L in the neighborhoods of all the crossing points of L (see Figure 34), so that the two points of the Legendrian knot, corresponding to the crossing point of L are antipodal in the fiber. After we

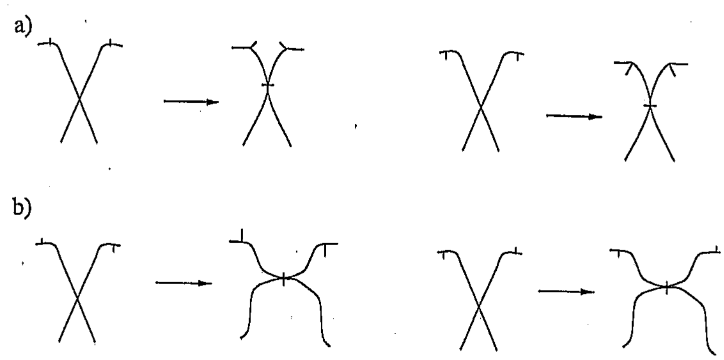

FIGURE 34. Making the preimages of the double point of L antipodal in ST^*F.

factorize the fibration by the $\mathbb{Z}_2$-action, the projection of the deformed λ is not a cooriented front anymore, but a front equipped with a normal field of lines. (This corresponds to the factorization $S^1 \to \mathbb{R}P^1$.) Using Figure 35, one calculates the inputs of different cusp and crossing points to the total rotation number of the line field, under traversing the boundary in the counter clockwise direction.

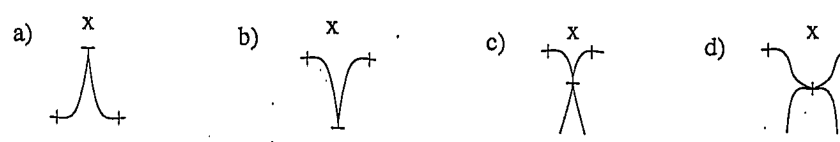

FIGURE 35. Inputs of crossing and cusp points into a gleam.

This inputs are:

$$\begin{cases} 1 & \text{for every cusp point, pointing inside } X \\ -1 & \text{for every cusp point, pointing outside } X \\ -1 & \text{for every crossing point of type, shown in Figure 11c} \\ 0 & \text{for the other types of crossing points} \end{cases} \tag{8.12}$$

To get the inputs to gleams, we divide these numbers by 2 (as we do in the construction of shadows, see 3.1).

If the region does not have any cusp, or crossing points in its boundary, then the obstruction to extend the section over ∂X to a section over X is equal to $\chi(\operatorname{Int} X)$.

If F is non-orientable, then to find the gleam of a region X, we first deform the front isotopically. So, that all the all the cusp and crossing points of ∂X are located very close to a boundary of a disc D, used to define shadows (see 3.2.J). Under this isotopy $\operatorname{gl}(X)$ does not change. Now it is clear, that the inputs of cusp and crossing points into $\operatorname{gl}(x)$ are the same as in the case, when F was oriented. As before, if X does not have cusp or crossing points on ∂X then the obstruction to extend a section over ∂X to a section over X is equal to $\chi(\operatorname{Int} X)$.

This finishes the proof of Theorem 6.2.C. □

8.11. Proof of Theorem 6.3.I. A straightforward check shows that

$$\operatorname{ind} \tilde{L}_u^+ - \operatorname{ind} \tilde{L}_u^- = \operatorname{ind} L_u^+ - \operatorname{ind} L_u^- - \epsilon(u) \tag{8.13}$$

$$\operatorname{ind} \tilde{L}_u^+ + \operatorname{ind} \tilde{L}_u^- = \operatorname{ind} L + 1 \tag{8.14}$$

$$\operatorname{ind} L_u^+ + \operatorname{ind} L_u^- = \operatorname{ind} L \tag{8.15}$$

for any double point u of L.

Let us prove prove identity (6.14). We write down the formal sums used to define $S(\lambda)$ and $l_q(\lambda)$ and start to reduce them in a parallel way, as it is described below.

We say that a double point u of L is essential, if $[\tilde{\lambda}_u^+] \neq [\tilde{\lambda}_u^-]$. If u is not essential, then we cancel out $[\tilde{\lambda}_u^+]$ and $[\tilde{\lambda}_u^-]$. Identity (8.13) implies, that for non-essential u $[\operatorname{ind} L_u^+ - \operatorname{ind} L_u^- - \epsilon(u)]_q = 0$.

The index of a wave front coincides with the homology class of its lifting under the natural identification of $H_1(ST^*F)$ with $\mathbb{Z}$. This and (8.14) implies, that if for two double points u and v $[\tilde{\lambda}_u^+] = [\tilde{\lambda}_v^-]$, then $[\tilde{\lambda}_u^-] = [\tilde{\lambda}_v^+]$. Hence, $[\tilde{\lambda}_u^+] - [\tilde{\lambda}_u^-] = -([\tilde{\lambda}_v^+] - [\tilde{\lambda}_v^-])$ and all these four terms cancel out. Identity (8.13) implies, that the terms $[\operatorname{ind} L_u^+ - \operatorname{ind} L_u^- - \epsilon(u)]_q$ and $[\operatorname{ind} L_v^+ - \operatorname{ind} L_v^- - \epsilon(v)]_q$ also cancel out.

Because of the similar reasons, if for a double point u the term $[\tilde{\lambda}_u^+] - [\tilde{\lambda}_u^-]$ is equal to $[\lambda] - f$ (so that these two terms cancel out with the term in front of C^+). Then $[\operatorname{ind} L_u^+ - \operatorname{ind} L_u^- - \epsilon(u)]_q = [\operatorname{ind} L - 1]_q$ and the input of this double point into $l_q(\lambda)$ also cancels out with a term in front of C^+

The same holds, if the input of a double point u into $S(\lambda)$ cancels out with a term in front of C^-.

Making the cancelations, described above, in both $S(\lambda)$ and $(l_q(\lambda) - [h]_q h)$ in a parallel way, we reduce $S(\lambda)$ as far as possible. After that we can not reduce further neither $S(\lambda)$ nor $l_q(\lambda)$. In this reduced form the terms of the form nf^m with $n > 0$ correspond to the $[\tilde{\lambda}_u^+]$ terms. Identities (8.13) and (8.14) imply that the input of the corresponding vertices into $l_q(\lambda)$ is $n[2m-h-1]_q$.

Of course a part of this nf^m could come form the cusp points, but the reader can easily check, that in this case the input of these cusps into $(l_q(\lambda) - [h]_q h)$ is also given by ϕ.

Thus, $l_q(\lambda) = \phi(S(\lambda)) + [h]_q h$ and we proved the identity (6.14).

Let us prove identity (6.15). We reduce $S(\lambda)$ and $(l_q(\lambda) - [h]_q h)$ in a parallel way, as above. After this reduction each term $n[m]_q$ is an input of n double points. (Note, that the coefficient in front of each $[m]_q$ was positive from the very beginning, because of the definition of $l_q(\lambda)$, and it stays positive under the cancelations described above.) Let u be one of these double points. Then from identities (8.13) and (8.14) we get a system of two equations in variables $\operatorname{ind}\tilde{L}_u^+$ and $\operatorname{ind}\tilde{L}_u^-$

$$\begin{cases} \operatorname{ind}\tilde{L}_u^+ - \operatorname{ind}\tilde{L}_u^- = m \\ \operatorname{ind}\tilde{L}_u^+ + \operatorname{ind}\tilde{L}_u^- = \operatorname{ind}L + 1 \end{cases} \tag{8.16}$$

Solving the system we get that that $[\tilde{\lambda}_u^+] = f^{\frac{m+h+1}{2}}$ and $[\tilde{\lambda}_u^-] = f^{\frac{h+1-m}{2}}$.

This proves identity (6.15) and we proved the Theorem. □

8.12. Proof of Theorem 7.2.A. There are five elementary isotopies of a generic front L on the orbifold F. Four of them are: the birth of the two cusps, passing through a non-dangerous self-tangency point, passing through a triple point and passing of a branch through a cusp point. For all the possible oriented versions of these moves a straightforward calculation shows that $S(\lambda) \in \frac{1}{2}\mathbb{Z}[H_1(N)]$ is preserved.

The fifth move is more complicated. It corresponds to a generic passing of a wave front lifted to $\mathbb{R}^2$ through the preimage of a cone point a. We can assume that this move is a symmetrization by G_a of the following move. The lifted front in the neighborhood of a is an arc C of a circle with a big radius centered on the y axis, and during this move this arc slides along the y axis through a (see Figure 36).

Clearly two points u and v of C after this move appear to be in the same fiber iff they are symmetric with respect to the y axis, and the angle formed by v, a, u is less or equal to π, and is equal to $\frac{2k\pi}{\mu_a}$ for some $k \in \{1, \dots, \mu_a\}$ (see Figure 36). We denote the set of such numbers k by $\bar{N}_1(a) = \{k \in \{1, \dots, \mu_a\} | \frac{2k\pi}{\mu_a} \in (0, \pi]\}$

They are in the same fiber before the move iff the angle formed by u, a, v is less than π and is equal $\frac{2k\pi}{\mu_a}$ for some $k \in \{1, \dots, \mu_a\}$ (see Figure 36). We denote the set of such numbers k by

$\bar{N}_2(a) = \{k \in \{1, \ldots, \mu_a\} \,|\, \frac{2k\pi}{\mu_a} \in (0, \pi)\}$

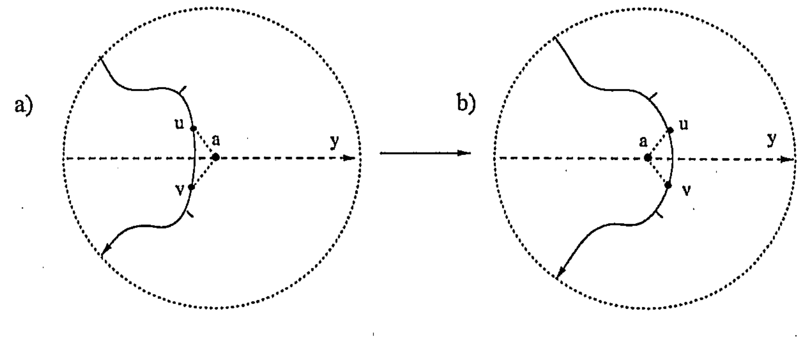

FIGURE 36. The Legendrian curve passes through the singular fiber.

The projection of this move for the orientation of L' depicted in Figure 36 is shown in Figure 37.

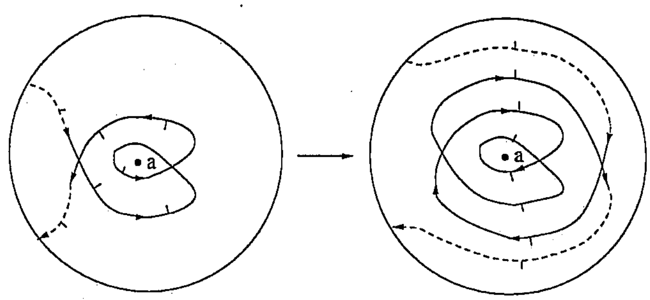

FIGURE 37. Projection of the Legendrian curve passing through the singular fiber.

Split the wave front in Figure 37 at the double point v (appearing after the move), corresponding to some $k \in \bar{N}_1(a)$. Then $\tilde{\lambda}_v^-$ is a front with two positive cusps that rotates k times around a in the clockwise direction. Hence, $[\tilde{\lambda}_v^-] = f f_a^{-k} = f_a^{\mu_a - k}$. We know that $[\tilde{\lambda}_v^+][\tilde{\lambda}_v^-] = [\lambda]f$ and that $f_a^{\mu_a} = f$. Thus, $[\tilde{\lambda}_v^+] = [\tilde{\lambda}]f_a^k$.

In the same way we check, that, if we split the front at the double point v (existing before the move), corresponding to some $k \in \bar{N}_2(a)$, then $[\tilde{\lambda}_v^+] = f_a^k$ and $[\tilde{\lambda}_v^-] = [K]f_a^{|\mu_a|-k}$.

Now, making sums over all the corresponding numbers $k \in \{1, \ldots, \mu_a\}$, we get that the jump of $S(\lambda)$ under this move is

$$\bar{R}_a^1 = \sum_{k \in \bar{N}_1(a)} ([\lambda]f_a^k - f_a^{\mu_a - k}) - \sum_{k \in \bar{N}_2(a)} (f_a^k - [\lambda]f_a^{\mu_a - k}). \tag{8.17}$$

A straightforward check shows that $R_a^1 = \bar{R}_a^1$. (Note that the sets $N_1(a)$ and $N_2(a)$ are different from $\bar{N}_1(a)$ and $\bar{N}_2(a)$.)

Thus the jump of $S(\lambda)$ is zero in J.

Taking the other orientation of C in this move, we get, that the corresponding jump of $S(\lambda)$ is equal to R_a^2.

Hence, the value of $S(\lambda)$ does not change under all the elementary isotopies, and we proved the theorem. □

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