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View-Obstruction Problems, III

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Suppose C is a closed convex body in E^n which contains the origin as an interior point. Define αC for each real number $\alpha \geqslant 0$ to be the magnification of C by the factor α and define $C + (m_1,..., m_n)$ for each point $(m_1,..., m_n)$ in E^n to be the translation of C by the vector $(m_1,..., m_n)$. Define the point set $\Delta(C, \alpha)$ by $\Delta(C, \alpha) = \{\alpha C + (m_1 + \frac{1}{2},..., m_n + \frac{1}{2})$: $m_1,..., m_n$ nonnegative integers$\}$. The *view-obstruction problem for* C is the problem of finding the constant $K(C)$ defined to be the lower bound of those α such that any half-line L given by $x_i = a_i t$ $(i = 1, 2,..., n)$, where the a_i $(1 \leqslant i \leqslant n)$ are positive real numbers and the parameter t runs through $[0, \infty)$, intersects $\Delta(C, \alpha)$. The paper considers the case where C is the n-dimensional cube with side 1, and in this case the constant $K(C)$ is evaluated for $n = 4$. The proof in dimension 4 depends on a theorem (proved via exponential sums) concerning the existence of solutions for a certain system of simultaneous congruences. The proofs in dimensions 2 and 3 are much simpler, and for these dimensions several other proofs have previously been given. For real x, let $\|x\|$ denote the distance from x to the nearest integer. A non-geometric description of our principal result is that we prove the case $n = 4$ of the following conjecture: For any n positive integers $w_1,..., w_n$ there is a real number x such that each $\|w_i x\| \geqslant (n+1)^{-1}$. © 1984 Academic Press, Inc.

1. INTRODUCTION

The view-obstruction problems defined in the abstract were first introduced in [2]. In this paper we only consider the case where the closed convex body C in E^n is the n-dimensional cube with side 1. We use the notation $\lambda(n)$ for the constant $K(C)$ in this case.

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For any real number x, let $\|x\|$ denote the distance from x to the nearest integer. The evaluation of $\lambda(n)$ can be thought of as a problem in Diophantine approximation, since we have

$$\tfrac{1}{2}\lambda(n) = \sup \min_{0 \leqslant x \leqslant 1} \max_{1 \leqslant i \leqslant n} \|w_i x - \tfrac{1}{2}\|, \tag{1}$$

where the supremum is taken over all n-tuples $w_1, ..., w_n$ of positive integers. Formula (1) follows from the definition of $\lambda(n)$ given in the abstract; we note that the positive real numbers a_i mentioned in the abstract can be assumed to be positive integers. If we define

$$\kappa(n) = \inf \max_{0 \leqslant x \leqslant 1} \min_{1 \leqslant i \leqslant n} \|w_i x\|, \tag{2}$$

where the infimum is taken over all n-tuples $w_1, ..., w_n$ of positive integers, then since $\|w_i x\| = \frac{1}{2} - \|w_i x - \frac{1}{2}\|$, we have $\lambda(n) = 1 - 2\kappa(n)$ for each $n \geqslant 2$. It will be convenient in the rest of the paper to concentrate on the problem of evaluating $\kappa(n)$.

The problem of evaluating $\lambda(n)$ is equivalent to the following: Suppose the unit cube in E^n has faces which reflect a certain particle, and consider any motion of the particle, *starting in a corner* of the cube and not entirely contained in a hyperplane of dimension $n-1$. What is the side length of the largest subcube, centered in the unit cube, with the property that there exists such a motion of the particle which does not intersect the subcube? Plainly the largest such side length is $\lambda(n)$.

The corresponding problem, if the condition that the particle start in a corner is omitted, can be treated by methods entirely different from those in this paper. This has been done by Schoenberg [5], who solved this problem in every dimension; he showed that the largest subcube in dimension n has side $1 - n^{-1}$.

The natural conjecture for the value of $\lambda(n)$ is $(n-1)/(n+1)$ (as stated in [2, p. 166]). This is because Dirichlet's box principle gives

$$\max_{0 \leqslant x \leqslant 1} \min_{1 \leqslant i \leqslant n} \|ix\| = \frac{1}{n+1},$$

so $\kappa(n) \leqslant 1/(n+1)$, and it is reasonable to conjecture that equality holds. That is, we conjecture that for any n positive integers $w_1, ..., w_n$, there is a real number x such that each $\|w_i x\| \geqslant (n+1)^{-1}$. The case $n = 2$ is very simple. The case $n = 3$ is more complicated, but several proofs have previously been published (Betke and Wills [1], Cusick [2–4]). The case $n = 4$ is solved here by an extension of the method of [4]. The proof in [4] was elementary, but the crucial step in the argument here is the estimation of certain exponential sums. The estimation succeeds only if a certain parameter is sufficiently large; dealing with the small values of the parameter requires some ad hoc calculations.

2. The Method of Proof

By (2), in order to show that $\kappa(n) = 1/(n+1)$ it is enough to prove that given any n-tuple $w_1,..., w_n$ of positive integers with the property that for any integers m and q,

$$\left\| w_i \frac{q}{m} \right\| \leqslant \frac{1}{n+1} \qquad \text{for some } i, \quad 1 \leqslant i \leqslant n, \tag{3}$$

there exists some pair m, q such that (3) does not hold if $\leqslant$ is replaced by $<$.

If we assume (as we may with no loss of generality) that $w_1,..., w_n$ have no common prime factor, then we would expect that there are only finitely many n-tuples $w_1,..., w_n$ such that (3) holds for any m and q. Further, we might hope that by considering only finitely many values of m, we could identify all of these n-tuples, and so reduce the determination of $\kappa(n)$ to a finite calculation. It is easy to carry out this procedure when $n = 2$, and so prove $\kappa(2) = 1/3$. When $n = 3$, the procedure can also be carried out; this was done in an elementary way in [4]. We apply this method for $n = 4$ in the following section, but the proof is no longer elementary. It is not clear whether the same method would be successful for $n \geqslant 5$, because of the increasing complexity of the various cases to which the problem would be reduced.

3. The Proof that $\kappa(4) = 1/5$

In this section, we take $n = 4$ and suppose w_1, w_2, w_3, w_4 are integers, having no common prime factor, such that (3) holds for any integers m and q. Our goal is to show that we can always find a pair of integers m and q such that

$$\min_{1 \leqslant i \leqslant 4} \left\| w_i \frac{q}{m} \right\| \geqslant \frac{1}{5}. \tag{4}$$

If w is not divisible by 5, then $\| w/5 \| \geqslant 1/5$, so we can assume that at least one of the w_i is divisible by 5. Thus there are several cases to consider, and it turns that the only difficult one is the case where exactly one of the w_i is divisible by 5. We dispose of the other cases first.

First suppose that $w_1 = 5^{i+k}a$, $w_2 = 5^{j+k}b$, $w_3 = 5^k c$, $w_4 = d$, where a, b, c, d are not divisible by 5 and $i \geqslant j \geqslant 0$, $k \geqslant 1$. We take $m = 5^{i+k+1}$ and will choose a q not divisible by 5, so $\|w_1 q/m\| \geqslant 1/5$. In order to specify q, we first choose a $q_0 \not\equiv 0 \bmod 5$ such that

$$bx \equiv t_1 \bmod 5^{i-j+1}, \qquad \|t_1/5^{i-j+1}\| \geqslant 1/5 \tag{5}$$

and

$$cx \equiv t_2 \bmod 5^{i+1}, \qquad \|t_2/5^{i+1}\| \geqslant 1/5 \tag{6}$$

both hold with $x = q_0$ for some choice of t_1, t_2. Such a q_0 exists because there are $3 \cdot 5^i + 5^j$ integers $x \bmod 5^{i+1}$ for which (5) holds for some t_1 and $3 \cdot 5^i + 1$ integers $x \bmod 5^{i+1}$ for which (6) holds for some t_2. Hence there are at least $5^i + 5^j + 1$ integers $x \bmod 5^{i+1}$ for which both (5) and (6) hold, and of these at least $5^j + 1$ are not divisible by 5. We define q to be $q_0 + 5^{i+1}r$, where r is chosen so that $\|w_4 q/m\| \geqslant 1/5$ (such a choice of r is possible since changing r by 1 changes $w_4 q/m$ by $d/5^k$). Clearly we have $\|w_2 q/m\|$ and $\|w_3 q/m\| \geqslant 1/5$ whatever choice of r is made, so (4) holds with the chosen q.

Now suppose that $w_1 = 5^{j+k}a$, $w_2 = 5^k b$, $w_3 = c$, $w_4 = d$, where a, b, c, d are not divisible by 5 and $j \geqslant 0$, $k \geqslant 1$. We take $m = 5^{j+k+1}$ and will choose a q not divisible by 5, so $\|w_1 q/m\| \geqslant 1/5$. In order to specify q, we first choose a $q_0 \not\equiv 0 \bmod 5$ such that $bq_0 \equiv t \bmod 5^{j+1}$, where t is an integer satisfying $\|t/5^{j+1}\| \geqslant 1/5$. There are $3 \cdot 5^j + 1$ such integers t, and so at least $2 \cdot 5^j + 1$ possible choices for $q_0 \not\equiv 0 \bmod 5$. We define q to be $q_0 + 5^{j+1}r$ where r is chosen so that both $\|cq/m\|$ and $\|dq/m\|$ are $\geqslant 1/5$. Such a choice of r is possible because both cq/m and dq/m run (in some order) through 5^k evenly spaced points mod 1 as r runs through $1, 2, \ldots, 5^k$. Thus we have $\|cq/m\| \geqslant 1/5$ for at least $3 \cdot 5^{k-1}$ values of r and $\|dq/m\| \geqslant 1/5$ for at least $3 \cdot 5^{k-1}$ values of r; hence for at least 5^{k-1} values of r, we have both inequalities. Plainly (4) holds for our choice of q.

Now suppose that $w_1 = 5^{k-1}a$, $w_2 = b$, $w_3 = c$, $w_4 = d$, where a, b, c, d are not divisible by 5 and $k \geqslant 2$. This is the only remaining case, and is the most difficult one. If we take $m = 5^k$, then (4) holds because of the following:

THEOREM. *Given any integer $k \geqslant 1$ and any integers b, c, d not divisible by 5, there exist integers t_1, t_2, t_3 and an integer q not divisible by 5 such that*

$$\begin{aligned} bq &\equiv t_1 \mod 5^k, \\ cq &\equiv t_2 \mod 5^k, \quad \left\| \frac{t_i}{5^k} \right\| \geqslant \frac{1}{5} \quad (i = 1, 2, 3). \\ dq &\equiv t_3 \mod 5^k, \end{aligned} \tag{7}$$

Thus the theorem implies our desired result that $\kappa(4) = 1/5$. The work below proves the theorem for each $k \geqslant 9$. The cases $k \leqslant 8$ can be handled by direct calculation. We are grateful to Mr. E. Abery for computer programming assistance in carrying out this calculation.

Let k be an integer with $k \geqslant 9$, let

$$I = \{i: 5^{k-1} \leqslant i \leqslant 4 \cdot 5^{k-1}\} \qquad \text{and let} \quad I_1 = \{i \in I: i \equiv 1 \bmod 5\}.$$

If r is an integer not divisible by 5, let $\mathcal{N}_k(r)$ denote the set of $q \in I_1$ such that $\|rq/5^k\| \geqslant 1/5$ and let $N_k(r)$ denote the cardinality of $\mathcal{N}_k(r)$.

In the theorem we can assume without loss of generality that $b = 1$. Thus the theorem follows from the assertion that if c, d are any integers not divisible by 5, then for each $k \geqslant 9$, $\mathcal{N}_k(c) \cap \mathcal{N}_k(d) \neq \varnothing$. Let $m = 5^k$. Since I_1 has exactly $.12m$ elements, it will follow that $\mathcal{N}_k(c) \cap \mathcal{N}_k(d)$ is non-empty if $N_k(c) + N_k(d) > .12m$. This is exactly what we will show except for a few choices of the pair c, d which we treat differently. Most of what we need is in the following two propositions.

PROPOSITION 1. *If r is such that there exist integers x, y with $|x|, |y| \leqslant 312$, $(x, y) = 1$, $(5, xy) = 1$, and $xr \equiv y \bmod m$, then $N_k(r) > .061m$, except that $N_k(4) = N_k(-4) = N_k(4^{-1} \bmod m) = N_k(-4^{-1} \bmod m) = .06m$.*

PROPOSITION 2. *If r is such that there do not exist integers x, y as described in Proposition 1, then $N_k(r) > .0601m$.*

To prove Proposition 1, we first reduce the estimation of an $N_k(r)$ to a finite calculation. Let J denote the set of real numbers z with $\|z\| \geqslant 1/5$. If S is a disjoint union of intervals, let $\mu(S)$ denote the sum of the lengths of these intervals.

LEMMA 1. *Suppose there exist positive integers x, y as described in Proposition 1. Then*

$$N_k(r) = \frac{m}{5y} \sum_{i=0}^{x-1} \mu\left\{\frac{y}{x}[i + .2, i + .8] \cap J\right\} + E(x, y)$$

where $|E(x, y)| < 1.6x + .6y$.

Proof. In what follows θ denotes a quantity of absolute value <1 and $\chi(S)$ denotes the number of connected components in the interior of the set S. We have

$$\begin{aligned} N_k(r) &= \#\left\{q \in [.2m, .8m]: q \equiv 1 \bmod 5, \frac{rq}{m} \in J\right\} \\ &= \sum_{j=0}^{x-1} \#\left\{q \in [.2m, .8m]: q \equiv 1 \bmod 5, q \equiv j \bmod x, \frac{rq}{m} \in J\right\} \\ &= \sum_{i=0}^{x-1} \#\left\{\frac{q}{m} \in [i + .2, i + .8]: q \equiv 1 \bmod 5, q \equiv 0 \bmod x, \frac{rq}{m} \in J\right\}. \end{aligned}$$

The last equality holds because there is an evident one-to-one correspondence between the jth summand in the first sum and the ith summand in the second sum if i and j satisfy $im \equiv -j \bmod x$. Now note that $q \equiv 0 \bmod x$, $(x, m) = 1$ and $xr \equiv y \bmod m$ imply $rq \equiv yq/x \bmod m$. Thus

$$\begin{aligned} N_k(r) &= \sum_{i=0}^{x-1} \# \left\{ \frac{q}{m} \in [i+.2, i+.8]: q \equiv 1 \bmod 5, q \equiv 0 \bmod x, \frac{yq}{xm} \in J \right\} \\ &= \frac{m}{5x} \sum_{i=0}^{x-1} \mu \left\{ [\mathrm{i}+.2, i+.8] \cap \frac{x}{y} J \right\} \\ &\quad + \theta \sum_{i=0}^{x-1} \chi \left\{ [i+.2, i+.8] \cap \frac{x}{y} J \right\} \\ &= \frac{m}{5y} \sum_{i=0}^{x-1} \mu \left\{ \frac{y}{x} [i+.2, i+.8] \cap J \right\} \\ &\quad + \theta \sum_{i=0}^{x-1} \chi \left\{ \frac{y}{x} [i+.2, i+.8] \cap J \right\}. \end{aligned}$$

Therefore

$$|E(x, y)| < \sum_{i=0}^{x-1} \left(.6 \frac{y}{x} + 1.6 \right) = 1.6x + .6y.$$

LEMMA 2. *If x, y are positive coprime integers, then*

$$\frac{1}{5y} \sum_{i=0}^{x-1} \mu \left\{ \frac{y}{x} [i+.2, i+.8] \cap J \right\} \geqslant .072 - \frac{.096}{x}.$$

Proof. Let $T = [.2(y/x), .8(y/x)]$. For each $\alpha \in T$, let $I(\alpha) = \{i \in \mathbb{Z}: 0 \leqslant i \leqslant x-1, \ \alpha + iy/x \in J\}$. Decompose T into disjoint intervals $T_1, T_2, \ldots, T_t$ such that $I(\alpha) = I_j$ is fixed for $\alpha \in T_j$. We have

$$\begin{aligned} \frac{1}{5y} \sum_{i=0}^{x-1} \mu \left\{ \frac{y}{x} [i+.2, i+.8] \cap J \right\} &= \frac{1}{5y} \sum_{i=0}^{x-1} \mu \left\{ \left(T + \frac{iy}{x} \right) \cap J \right\} \\ &= \frac{1}{5y} \sum_{j=1}^{t} \sum_{i=0}^{x-1} \mu \left\{ \left(T_j + \frac{iy}{x} \right) \cap J \right\} \\ &= \frac{1}{5y} \sum_{j=1}^{t} \sum_{i \in I_j} \mu \left(T_j + \frac{iy}{x} \right) \\ &= \frac{1}{5y} \sum_{j=1}^{t} \mu(T_j) \cdot \# I_j. \end{aligned}$$

Now any $\# I(\alpha)$ is $\geqslant [.6x]$. To see this, note that (here $\{x\}$ denotes the fractional part of x)

$$\left\{ \left\{ \alpha + \frac{iy}{x} \right\} : 0 \leqslant i \leqslant x-1 \right\} = \left\{ \left\{ \alpha + \frac{i}{x} \right\} : 0 \leqslant i \leqslant x-1 \right\}$$

since gcd $(x, y) = 1$. Furthermore

$$\left\{ \left\{ \alpha + \frac{i}{x} \right\} : 0 \leqslant i \leqslant x-1 \right\} \cap [.2, .8]$$

consists of $\geqslant [.6x]$ equally spaced points of common gap $1/x$. Thus

$$\frac{1}{5y} \sum_{i=0}^{x-1} \mu \left\{ \frac{y}{x} [i+.2, i+.8] \cap J \right\} \geqslant \frac{[.6x]}{5y} \sum_{j=1}^{t} \mu(T_j)$$
$$= \frac{[.6x]}{5y} \left(\frac{.6y}{x} \right) \geqslant \frac{(.6x-.8)(.6)}{5x} = .072 - \frac{.096}{x}.$$

Proof of Proposition 1. Since $N_k(r) = N_k(-r) = N_k(r^{-1} \bmod m)$, we may assume x, y are positive integers with $1 \leqslant x \leqslant y \leqslant 312$. With the kind assistance of D. E. Penney we have directly calculated the sum in Lemma 1 for each pair x, y with $x \leqslant 9$. In each case, except for $x = 1, y = 4$, we have the sum at least $(1/15)m$. Since $|E(x, y)| < 200$ and $m \geqslant 5^9$, we have $N_k(r) > .061m$ in each case except $r \equiv \pm 4, \pm 4^{-1} \bmod m$. Working through the proof of Lemma 1 in the case $x = 1, y = 4$, we see that $E(1, 4) = 0$ and that $N_k(4) = .06m$.

Now assume $x \geqslant 10$. Then from Lemmas 1 and 2

$$N_k(r) > \left(.072 - \frac{.096}{10} \right) m - (1.6)(311) - (.6)(312)$$
$$> .0624m - 685 > .061m,$$

since $m \geqslant 5^9$.

Proof of Proposition 2. Fix an integer r not divisible by 5 for which there does not exist a pair x, y as described in Proposition 1. Let $|t|_m$ denote the absolute value of the residue of $t \bmod m$ that is closest to 0. Thus there is no integer t not divisible by 5 such that both $|t|_m$ and $|rt|_m$ are less than 313.

Let $e(x) = e^{2\pi i x}$. We have

$$N_k(r) = \frac{1}{m} \sum_{t=0}^{m-1} \sum_{q \in I} \sum_{p \in I_1} e \left(\frac{t(q-rp)}{m} \right)$$
$$= \frac{1}{m} (.6m+1)(.12m) + \frac{1}{m} \sum_{t=1}^{m-1} \left(\sum_{q \in I} e \left(\frac{tq}{m} \right) \right) \left(\sum_{p \in I_1} e \left(\frac{-trp}{m} \right) \right). \qquad (8)$$

Summing the geometric progressions in the inner sums we have

$$\left|\sum_{q\in I} e\left(\frac{tq}{m}\right)\right| \leqslant \frac{1}{2\,\|t/m\|}$$

$$\left|\sum_{p\in I_1} e\left(\frac{-trp}{m}\right)\right| \leqslant \frac{1}{2\,\|5rt/m\|}.$$

The main term on the right of (8) is $.072m$. The error term is bounded in absolute value by

$$\begin{aligned} .12+\frac{1}{m}\sum_{t=1}^{m-1}\frac{1}{2\,\|t/m\|}\cdot\frac{1}{2\,\|5rt/m\|} &= .12+\frac{m}{4}\sum_{t=1}^{m-1}\frac{1}{|t|_m}\cdot\frac{1}{|5rt|_m} \\ &= .12+\frac{m}{2}\sum_{t=1}^{(m-1)/2}\frac{1}{t}\cdot\frac{1}{|5rt|_m} \end{aligned}$$

since $|5rt|_m = |5r(m-t)|_m$, $|t|_m = |m-t|_m$.

We consider 4 cases to estimate the last sum.

Case 1. $t \leqslant 312$, $5 \nmid t$. Then $|rt|_m > 312$, so $|5rt|_m \geqslant 1565$. Thus the portion of the sum in this case is

$$\leqslant \frac{m}{2}\cdot\frac{1}{1565}\sum_{\substack{t=1\\5\nmid t}}^{312}\frac{1}{t} < .00172m.$$

Case 2. $t \leqslant 312$, $5 \mid t$. Note that the map $t \in [1,(m-1)/2] \mapsto |5rt|_m$ is 5: 1. It is 1: 1 on the restricted domain $[1,(m/5-1)/2]$. If t is in this restricted domain, then the other values that map to $|5rt|_m$ are $m/5-t$, $m/5+t$, $2m/5-t$, $2m/5+t$.

For $t \leqslant 312 \leqslant (m/5-1)/2$, the values of $|5rt|_m$ are distinct, and since $5 \mid t$, the values $|5rt|_m$ are divisible by 25. Thus the portion of the sum in this case is

$$\leqslant \frac{m}{2}\sum_{t=1}^{62}\frac{1}{5t}\cdot\frac{1}{25t} = \frac{m}{250}\left(\frac{\pi^2}{6}-\sum_{t=63}^{\infty}\frac{1}{t^2}\right)$$

$$< \frac{m}{250}\left(\frac{\pi^2}{6}-\frac{1}{63}\right) < .00652m.$$

Case 3. $t > 312$, $|rt|_m \leqslant 312$. Considering the 5 choices of t corresponding to each value of $|5rt|_m$, the portion of the sum in this case is (using $m \geqslant 5^9$)

$$\leqslant \frac{m}{2}\left(\frac{1}{313}+\frac{1}{m/5-313}+\frac{1}{m/5+313}+\frac{1}{2m/5-313}+\frac{1}{2m/5+313}\right)$$

$$\times \sum_{t=1}^{312}\frac{1}{5t} < .00203m.$$

Case 4. $t > 312$, $|rt|_m > 312$. Again, for each value of $|5rt|_m$, there are 5 values of t. The value of t which can do the most damage, of course, is the smallest. Thus the portion of the sum in this case is

$$\leqslant 5 \cdot \frac{m}{2} \sum_{t=313}^{(m/5-1)/2} \frac{1}{t} \cdot \frac{1}{5t} < \frac{m}{2} \sum_{t=313}^{\infty} \frac{1}{t^2} < \frac{m}{624} < .00161m.$$

Finally we note that $.12 \leqslant (.12/5^9)\, m < 10^{-7}m$, so that the absolute value of the error term on the right of (8) is $< .0119m$. Thus $N_k(r) > .0601m$.

Proof of the Theorem. We need to show that if c, d are integers not divisible by 5, then $\mathscr{N}_k(c) \cap \mathscr{N}_k(d) \neq \varnothing$. Except for the case when both c, d are found in the set $\{\pm 4 \bmod m,\ \pm 4^{-1} \bmod m\}$, Propositions 1 and 2 show that $N_k(c) + N_k(d) > .12m$, so that as noted above, $\mathscr{N}_k(c) \cap \mathscr{N}_k(d) \neq \varnothing$.

Since $\mathscr{N}_k(r) = \mathscr{N}_k(-r)$, to complete the proof we need only show that $\mathscr{N}_k(4) \cap \mathscr{N}_k(4^{-1} \bmod m) \neq \varnothing$. To see this, let q denote the first integer above $\frac{2}{5}m$ with $q \equiv 3 \bmod 4$ and $q \equiv 1 \bmod 5$. That is, $q = \frac{2}{5}m + 1$. Then $q \in \mathscr{N}_k(4) \cap \mathscr{N}_k(4^{-1} \bmod m)$ since $q \in I_1$, $\|4q/m\| \approx \frac{2}{5}$, and

$$\left\| \frac{(4^{-1} \bmod m)\, q}{m} \right\| = \left\| \frac{(q+m)/4}{m} \right\| \approx \frac{7}{20}.$$

4. Concluding Remarks

We have not discussed the problem of explicitly determining all the sets $\{w_1, w_2, \ldots, w_n\}$ for which the max min in (2) is equal to $1/(n+1)$. It is known (see [2, pp. 169–170] and [3, p. 11]) that for $n = 2$ or 3 the only such sets are the obvious ones $\{k, 2k, \ldots, nk\}$, where k is some positive integer. The situation is certainly not this simple if $n \geqslant 4$; for example, the max min in (2) is equal to $1/5$ if $\{w_1, w_2, w_3, w_4\} = \{1, 3, 4, 7\}$ and is equal to $1/6$ if $\{w_1, w_2, \ldots, w_5\} = \{1, 3, 4, 5, 9\}$. Perhaps this has something to do with the apparent difficulty in finding an elementary approach to the problem if $n \geqslant 4$.

References

1. U. Betke and J. M. Wills, Untere Schranken für zwei diophantische Approximations-Funktionen, *Monatsh. Math.* **76** (1972), 214–217.
2. T. W. Cusick, View-obstruction problems, *Aequationes Math.* **9** (1973), 165–170.
3. T. W. Cusick, View-obstruction problems in n-dimensional geometry, *J. Combin. Theory Ser. A* **16** (1974), 1–11.
4. T. W. Cusick, View-obstruction problems, II, *Proc. Amer. Math. Soc.* **84** (1982), 25–28.
5. I. J. Schoenberg, Extremum problems for the motions of a billiard ball, II. The L_∞ norm, *Nederl. Akad. Wetensch. Proc. Ser. A* **79** *Indag. Math.* **38** (1976), 263–279.