

# Accurate Evaluation of Layer Potentials up to the Boundary



## Introduction

### DIRICHLET BOUNDARY VALUE PROBLEM

The goal of our research is to estimate the numerical solutions to the Laplace's equation, i.e. given a domain  $\Omega$  and its boundary  $\partial\Omega$ , find a continuous function  $u$  satisfying the following equations

$$\begin{cases} \Delta u = u_{xx} + u_{yy} = 0 \text{ in } \Omega \\ u = f \text{ on } \partial\Omega \text{ with a given function } f \end{cases}$$

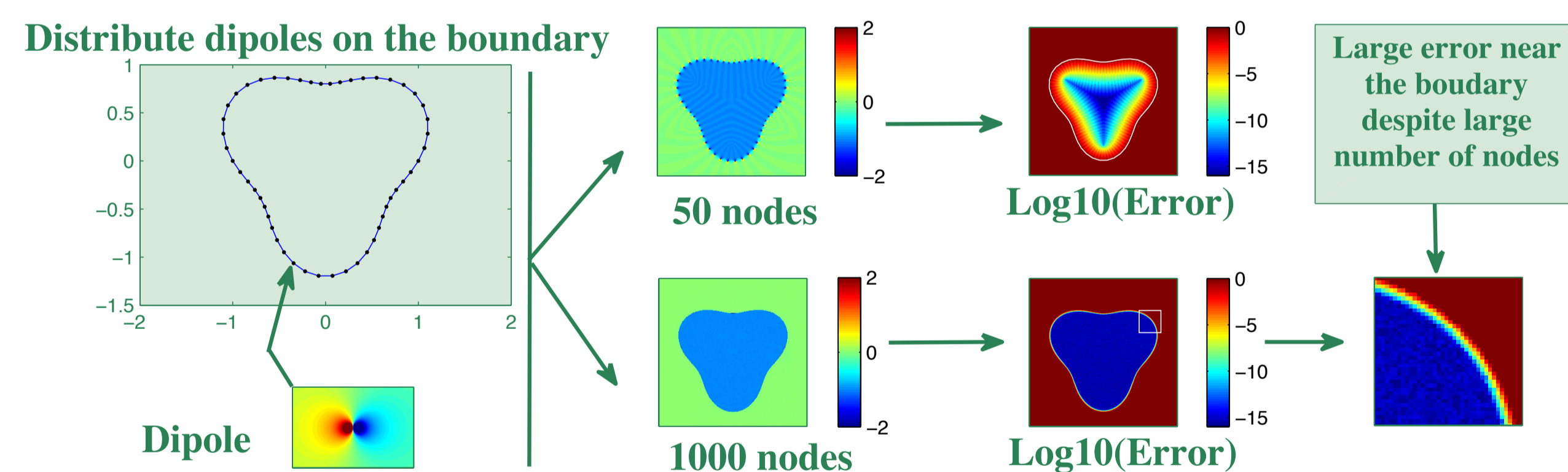
Applying potential theory [1], we represent the solution uniquely as a double-layer potential where  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$ :

$$u(\mathbf{x}) = \int_{\partial\Omega} \frac{\partial\Phi(\mathbf{x}, \mathbf{y})}{\partial\mathbf{n}_y} \tau(\mathbf{y}) ds_y \quad \text{where} \quad \Phi(\mathbf{x}, \mathbf{y}) = \frac{1}{2\pi} \log \frac{1}{|\mathbf{x} - \mathbf{y}|}$$

For a given density function  $\tau(\mathbf{y})$  on the boundary, we need to find an algorithm to evaluate  $u$  in the interior as accurately as possible.

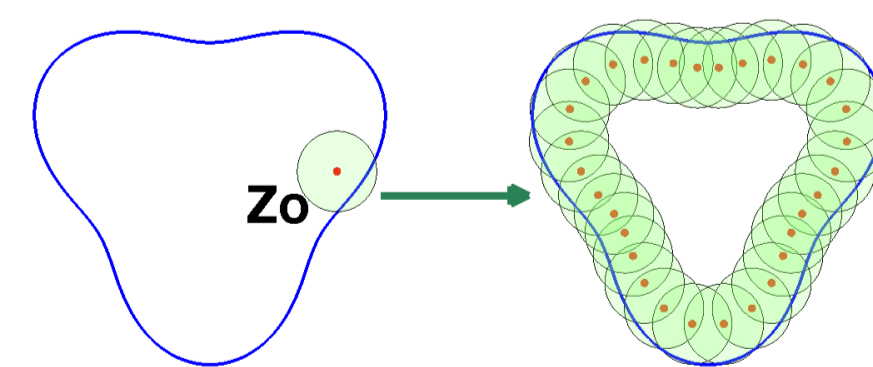
### ERROR NEAR THE BOUNDARY

A prevalent method to evaluate  $u$  uses *discrete quadrature* to turn the integral into the sum of the *dipoles* on  $\partial\Omega$ , which is accurate on the inner region but yields unacceptable error near  $\partial\Omega$ .



### TAYLOR EXPANSION

To resolve the aforementioned problem, we identify  $\mathbb{R}^2$  with  $\mathbb{C}$  and use the *Taylor expansion* centered at a point  $z_0$  near  $\partial\Omega$  to evaluate  $u$  locally instead of using quadrature on the entire domain.



$$u(\mathbf{x}) = \text{Re} \left( \sum_{n=0}^T c_n (z - z_0)^n \right) \quad \text{where} \quad \begin{cases} z = x_1 + ix_2 \\ \mathbf{x} = (x_1, x_2) \end{cases}$$

To evaluate the coefficient  $c_n$ , we introduce a boundary integral and Lagrange Interpolation. Now, from a small number of given values of  $u$  on  $\partial\Omega$ , say  $N$ , we can evaluate  $u$  accurately up to the boundary.

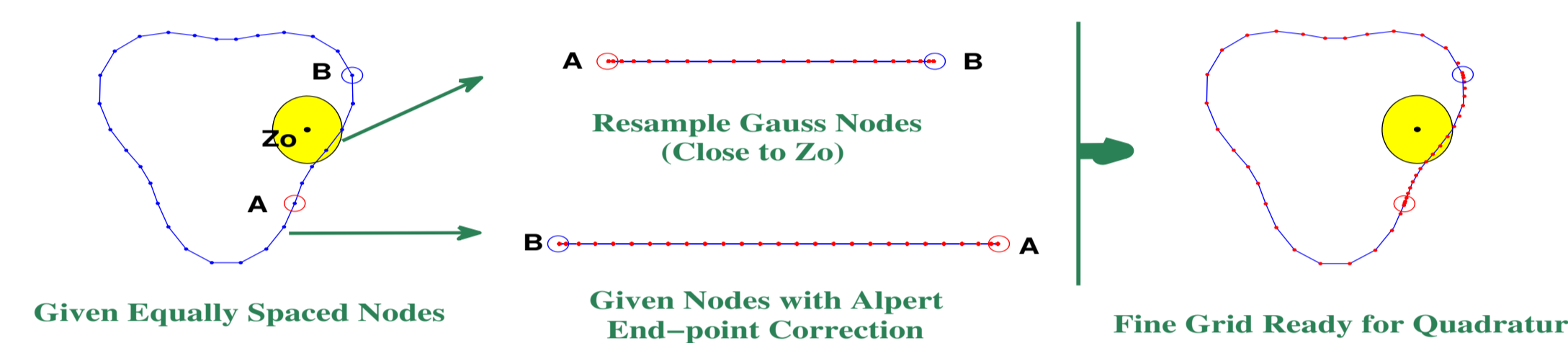
## Evaluating the Taylor Coefficients

### ALPERT CORRECTION & GAUSSIAN QUADRATURE

We can express the  $n^{\text{th}}$  Taylor coefficient as the following integral where  $y(t), 0 \leq t < 2\pi$  parametrizes  $\partial\Omega$  in the complex plane

$$c_n = \frac{-1}{2\pi} \int_0^{2\pi} \frac{e^{i\gamma(y(t))}}{(y(t) - z_0)^{n+1}} \tau(y(t)) y'(t) dt$$

To evaluate  $c_n$ , we build a set of finely spaced nodes on  $\partial\Omega$  consisting of Gauss nodes on  $AB$  and original nodes on  $BA$  with the endpoints handled by Alpert correction.[2]



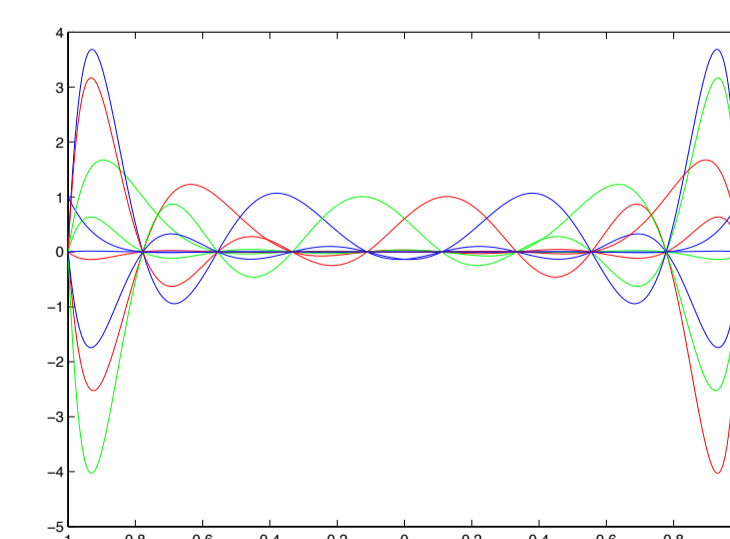
With the new grid, we apply Gaussian quadrature to estimate

$$\int_0^{2\pi} F(t) dt \approx \sum_{i=1}^n w_i F(t_i) \quad \text{where } w_i \text{ are known weights.}$$

Gaussian quadrature is guaranteed to converge uniformly for any continuous function.

### LAGRANGE INTERPOLATION

With the new grid, we need to evaluate  $\tau(t)$  at *any* point  $t$  given only the set of  $N$  points. Taking  $n$  local points  $t_i$  of density  $\tau_i, i \in \{1, 2, \dots, n\}$ , we want to find a polynomial  $p(t)$  that approximates  $\tau(t)$ , i.e. for all  $i, p(t_i) = \tau_i$ .



Lagrange polynomials on  $[-1, 1]$  for  $n = 10$

First, we define *Lagrange polynomials*:

$$L_j(t) = \prod_{i=1, i \neq j}^n \frac{t - t_i}{t_j - t_i} \quad \text{so} \quad L_j(t_i) = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

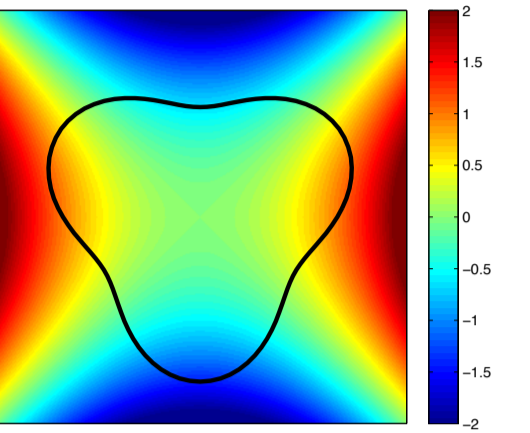
The interpolating polynomial can be written as

$$p(t) = \tau_1 L_1(t) + \tau_2 L_2(t) + \dots + \tau_n L_n(t)$$

The equally spaced interpolation can sometimes diverge on the edges of the interval (e.g. *Runge's Phenomenon*). In our algorithm, we choose  $t_i$  such that the value of interest is at the middle of the interval where the interpolation is guaranteed to converge.

## Result

We test our algorithm with a known function  $u = x^2 - y^2$  and use *MPSpack* [3] to generate a density function  $\tau$  for  $N = 100$  points on the boundary. Parameters for our evaluation method include:

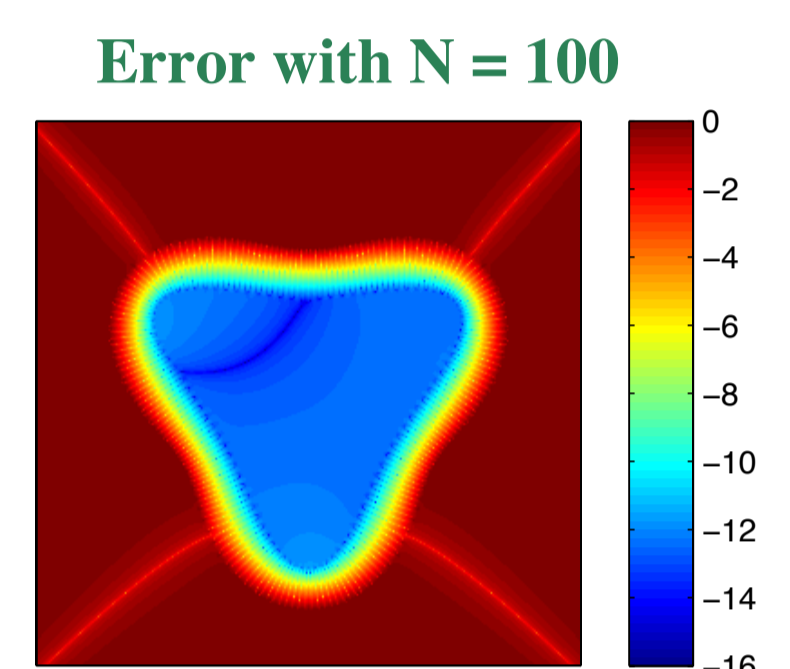
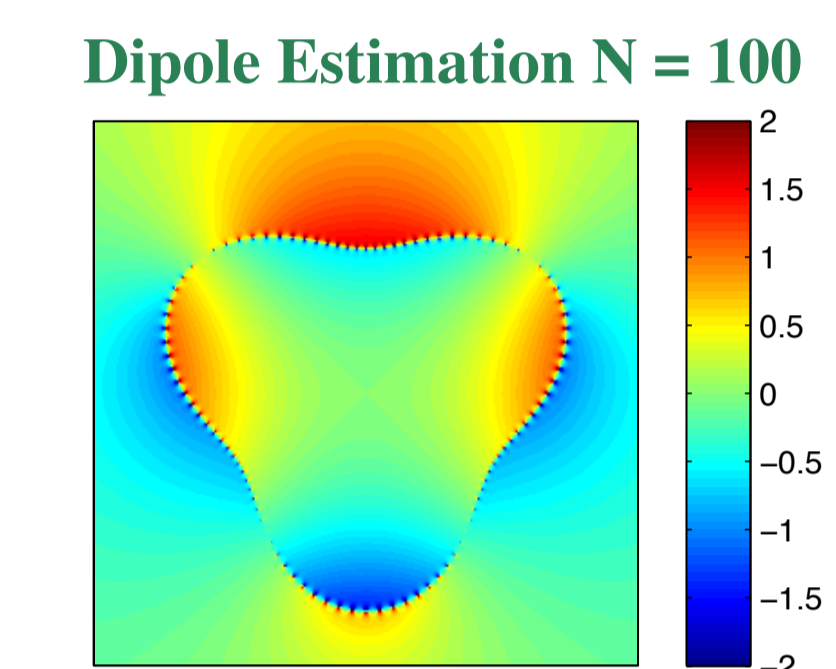


$$u = x^2 - y^2$$

- $|z_0|$ : radius of the Taylor expansion center
- $\delta$ : radius of the Taylor discs;
- $N_{z_0}$ : number of points  $z_0$ .

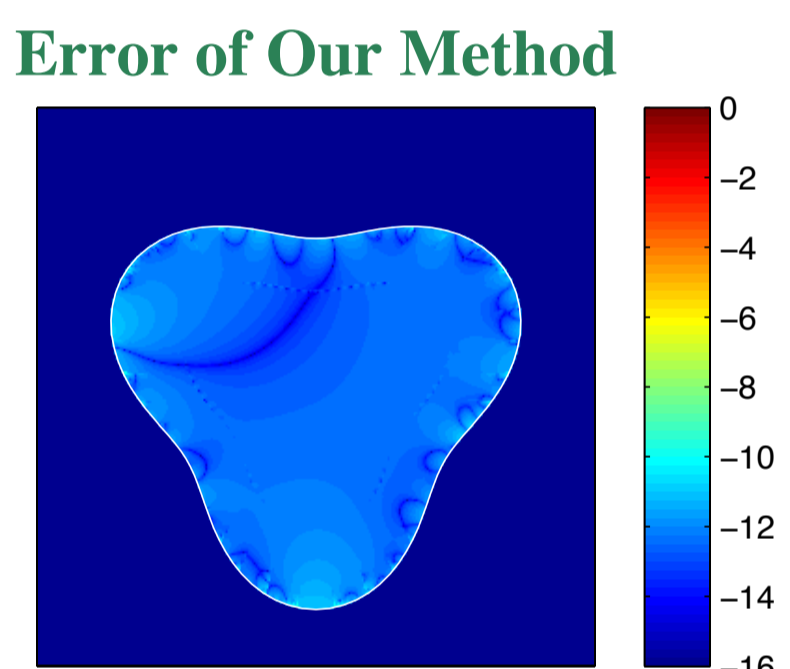
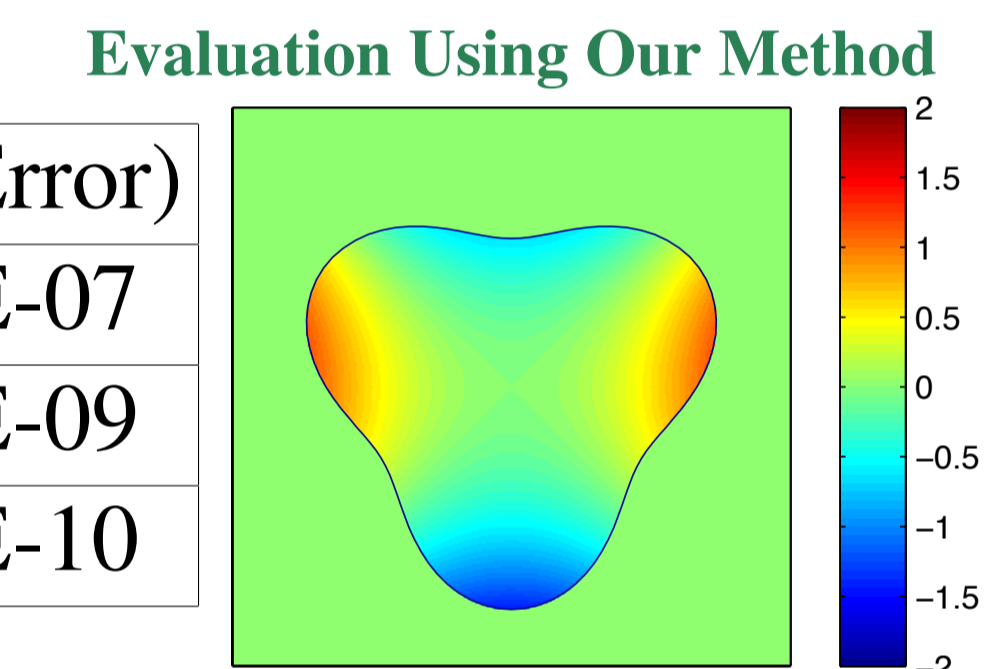
### Conventional Method

$ z $	Error
0.6	8.55E-13
0.7	6.44E-11
0.8	2.43E-07
0.9	7.80E-04
0.95	4.11E-02



### Our Method

$\delta$	$ z_0 $	$N_{z_0}$	Max(Error)
0.15	0.7	30	2.48E-07
0.2	0.7	30	5.98E-09
0.2	0.7	40	6.52E-10



Notes:

- $\log_{10}$  (Error) is shown.
- All trials use Taylor degree  $T = 30$ , Gaussian quadrature with 200 points, interpolation order  $P = 20$ , and Alpert order  $A = 20$ .

## Conclusion

Combining the mentioned techniques, we derive an algorithm to evaluate the numerical solutions of the Dirichlet Boundary Value Problem with high accuracy up to the boundary from a small number of given nodes. To ensure its accuracy, we need to test our method further on different  $u$  with singularities outside but near  $\Omega$ .

## References

- [1] Kress, Rainer. Linear Integral Equations. Berlin: Springer-Verlag, 1989. Print.
- [2] Alpert, Bradley. "High-order Quadratures for Integral Operators with Singular Kernels." Journal of Computational and Applied Mathematics 60.3 (1995): 367-78. Print.
- [3] Barnett, Alex, and Timo Betcke. MPSpack. MATLAB Toolbox to solve Helmholtz PDE, wave scattering, and eigenvalue problems.