V63.0140-3: Linear Algebra. QUIZ 4 SOLUTIONS

1. Given the matrix $A = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$,

   (a) $\det(A - \lambda I) = (4 - \lambda)(1 - \lambda) + 2 = (\lambda - 3)(\lambda - 2)$. Eigenvalues are $\lambda = 2, 3$.

   (b) $\lambda = 2$: find null-space vectors in $(A - 2I)$.

   $\lambda = 3$: find null-space vectors in $(A - 3I)$.

   Each time, you should find one free variable. (If you don't find any, you know λ is not a correct eigenvalue).

   $$\mathbf{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

   (c) $P =$ eigenvectors stacked as columns. $D =$ eigenvalues. Note you must match up the order of eigenvectors with eigenvalues. You must also find the inverse of P - be carful with signs since $\det P$ is either $+1$ or -1 depending on your choice of eigenvectors. For example,

   $$A = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}.$$

2. (a) It can be diagonalizable, but only if the dimension of the eigenspace corresponding to the (multiplicity 2) eigenvalue $\lambda = -2$ is also 2. Then 3 lin. indep. eigenvectors exist.

   (b) The original vector's length is $\sqrt{1^2 + 3} = 2$. So divide both entries by this length: $\begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix}$. This vector now is a unit vector.

   (c) Always true, since each distinct eigenvalue must have an eigenvector, and we showed this set is lin. indep., so can be used for diagonalization. I think some of you thought that for B an $n \times n$

matrix, if there were less than n real eigenvalues, then B is not diagonalizable. This is not true. Counting complex eigenvalues, there are always n eigenvalues (roots of characteristic equation). If they are distinct, then the above follows.

(d) Eigenvectors found via $(A - \lambda I)\mathbf{v} = \mathbf{0}$:
$\begin{bmatrix} -1-3i & 5 \\ -2 & 1-3i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Since there will be one free variable (because a 2×2 matrix has two eigenvalues, and the other one must therefore be the conjugate, $2 - 3i$), you can use *either* row to give the vector (as in book). This trick only works for the 2×2 case.

e.g. second row: $-2v_1 + (1-3i)v_2 = 0$ gives $\mathbf{v} = \begin{bmatrix} 1-3i \\ 2 \end{bmatrix}$. Careful with signs!

$\mathbf{v} = \begin{bmatrix} 5 \\ 1+3i \end{bmatrix}$ is also correct (they are parallel, even though they don't look it! The complex arithmetic to show this is not hard).

(e) It is a repellor because both eigenvalues are larger than 1. (Strictly, have magnitudes larger than 1). How to remember? Taking higher powers of numbers which are larger than 1 results in heading off to infinity, *i.e.* repelling away from the origin.

3. (a) Inner product $\mathbf{u} \cdot \mathbf{y} = 10$, which is not 0, so they are not orthogonal.

(b) orthogonal projection

$$\hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}} \mathbf{u} = \frac{10}{5} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

. This is the vector you get by dropping a perpendicular from the location $\mathbf{y}$ to the direction given by $\mathbf{u}$. Note that the answer is a scalar multiple of $\mathbf{u}$ (not $\mathbf{y}$), even though it is called $m\hat{b}fy$. It's easy to remember this when you draw the projection diagram.