V63.0140-3: Linear Algebra. QUIZ 3 SOLUTIONS

1. A basis for the column space of A is the pivot columns 1, 3, 4. Be sure to use the entries in the original A not the R.E.F.!

$$\left\{ \begin{bmatrix} -2 \\ 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 1 \\ -2 \end{bmatrix} \right\}.$$

The null space is the solution set of $A\mathbf{x} = \mathbf{0}$, and a basis is the vectors you get from the two free variables x_2 and x_5:

$$\left.\begin{array}{rcrr} x_1 & = & -2x_2 & +x_5 \\ x_2 & = & x_2 & \\ x_3 & = & & -2x_5 \\ x_4 & = & & -x_5 \\ x_5 & = & & x_5 \end{array}\right\} \text{ so basis is } \{\mathbf{u}, \mathbf{v}\} = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ -1 \\ 1 \end{bmatrix} \right\}$$

where we wrote the solution set as $\mathbf{x} = s\mathbf{u} + t\mathbf{v}$.

dim Nul A = number of basis vectors in Nul A = number of free vars = 2.

Rank A = dim Col A = number of pivot columns = 3.

The dimension of the subspace formed by the span of the rows of A just means dim Row A, which = dim Col A = rank A by the Rank Theorem. So it is also 3.

2. (a) False: $\mathbb{R}^2$ is not a subspace of $\mathbb{R}^3$ because its elements (2-component vectors) do not even come from $\mathbb{R}^3$ (the set of 3-component vectors). It is not even a subset.

   (b) True: The null space of A from question 1 involves $\mathbf{x}$ vectors which have 5 components, so are a subset of $\mathbb{R}^5$. A null space is always a subspace.

   (c) False: Col A from question 1 has dimension 3, so is isomorphic to $\mathbb{R}^3$. It lives in $\mathbb{R}^4$ though (it is a subspace of $\mathbb{R}^4$).

(d) Write the 3 polynomials in the standard basis, to get

$$\left\{ \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\} \text{ stack as cols, reduce to get } \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & -1 \end{bmatrix}$$

All 3 pivots, so we have 3 linearly-independent vectors in $\mathbb{R}^3$, so they form a basis. (You could also instead have said they span $\mathbb{R}^3$). $\mathbb{P}_2$ is isomorphic to $\mathbb{R}^3$ so the original polynomials also form a basis for $\mathbb{P}_2$.

(e) H is a subset of $\mathbb{R}^2$, but a) does not include the $\mathbf{0}$ vector, b) is not closed under addition, and c) is not closed under scalar multiplication. So H is not a subspace.

3. There are 2 basis vectors, so you know $[\mathbf{x}]_{\mathcal{B}}$ must have 2 components, call them c_1 and c_2.

$$c_1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ -5 \end{bmatrix}$$

must be satisfied, since this what the $\mathcal{B}$-coords of $\mathbf{x}$ mean. This is just a linear equation which we solve by row reduction of augmented matrix:

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 4 \\ -1 & 1 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{(R.E.F.)}$$

It is consistent (otherwise $\mathbf{x}$ would not be in H). The unique solution is $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$.