V63.0140-3: Linear Algebra. QUIZ 2 Solutions

Many of you didn't heed this: "give explanations wherever you can". It helps you get points (or show me that you're confused)

1. [3 points]

   REF is $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ giving

$$\begin{aligned} x_1 &= 3 - 2x_2 \\ x_2 &= \text{free} \end{aligned}$$

   which is acceptable as parametric form (x_2 being the parameter). Remember only free vars can be parameters. A is 2×2 so there are only 2 variables (no x_3 !) Really I intended you to continue and write as

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

   where t is the parameter (we just renamed x_2, as is standard).

2. [6 points]

   There is a standard procedure which you should be familiar with by now: stack the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ as the columns of matrix A, and if there are columns missing pivots (*i.e.* free vars) then $A\mathbf{x} = \mathbf{0}$ can have non-trivial solutions, and the vectors are linearly dependent. This happens thus:

$$A = \begin{bmatrix} 1 & 7 & 3 \\ 2 & 0 & -1 \\ 0 & 2 & 1 \\ 1 & 3 & 1 \end{bmatrix} \sim \begin{bmatrix} \underline{1} & 7 & 3 \\ 0 & \underline{2} & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{(E.F.)} \quad \sim \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{(R.E.F.)}$$

Last col (x_3) has no pivot so is free var (pivots are underlined), therefore the answer is no: vectors are linearly dependent. [Do *not* get confused by thinking that A is an augmented matrix and discussing consistency—this is unrelated and wrong!]

The linear dependence relation means write some non-trivial linear combination of the vectors gives zero. This is same as looking for non-trivial solutions $\mathbf{x}$ to $A\mathbf{x} = \mathbf{0}$. So we read off from R.E.F. (you *can't* do this from E.F.) the parameteric vector form of solution,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}x_3 \\ -\frac{1}{2}x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix} = t \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}.$$

If it helps you, you can put an imaginary column of zeros as the "**b**" on the RHS of the augmented matrix as you do this.

Note we chose t to make the numbers integers in the final vector (like the chemistry application, it's easier). This is the same as the linear dependence relation, our answer:

$$\mathbf{v}_1 - \mathbf{v}_2 + 2\mathbf{v}_3 = \mathbf{0}.$$

3. [6 points]

   (a) True. Commute $\Rightarrow AB = BA \Rightarrow AB - BA = 0$, (0 is the zero matrix, to be strict).

   (b) False. If the range of a transformation T does not fill the codomain then T must not be onto the codomain. This is definition of 'onto'. One-to-one is an unrelated issue (do not get confused by Invertible Matrix Theorem which only applies to *square* matrices).

   (c) The vector we want to transform is $\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Linear transformations obey (by definition)

$$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$$

so we can sum the transformed vectors to get the transform of the sum: $\begin{bmatrix} 3 \\ 4 \end{bmatrix} + \begin{bmatrix} -5 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ is the answer.

(d) Special simple formula since it's 2×2. Invertible if $ad - bc \neq 0$. In our case $2.6 - (-3).(-4) = 0$ so it's singular, *i.e.* not invertible.

4. [5 points]

Try the rotation on the two basis vectors $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

$T(\mathbf{e}_1) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ and $T(\mathbf{e}_2) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Stack these two in the correct order to get

$$A = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Check your signs by trying out on a vector you know.

T is onto $\mathbb{R}^2$ because of many reasons,

(a) T is a rotation so clearly every point in $\mathbb{R}^2$ is the image of some $\mathbf{x}$, or

(b) A has a pivot in every column so the columns span $\mathbb{R}^2$, or

(c) A is square and invertible (by Invertible Matrix Theorem).

You don't need to use any fancy derivation using $\sin\theta$ etc—I didn't ask for this.