V63.0140-3: Linear Algebra. QUIZ 1 Solutions

1. Write augmented matrix and row reduce:

$$\begin{array}{rrrrcr} & & 3x_3 + & x_4 & = & 1 \\ 2x_1 + & 6x_2 + & x_3 - & 2x_4 & = & 15 \\ -x_1 - & 3x_2 + & 2x_3 + & x_4 & = & -5 \end{array}$$

$$\begin{bmatrix} 0 & 0 & 3 & 1 & 1 \\ 2 & 6 & 1 & -2 & 15 \\ -1 & -3 & 2 & 1 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -2 & -1 & 5 \\ 0 & 0 & 5 & 0 & 5 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix} \quad \textit{e.g.}\text{ echelon form}$$

$$\sim \begin{bmatrix} 1 & 3 & 0 & 0 & 5 \\ & & 1 & 0 & 1 \\ & & & 1 & -2 \end{bmatrix} \quad \text{reduced echelon form (unique)}$$

Note the question *asks* you to go to reduced echelon form (R.E.F.)

Consistent since no impossible row of the form $[0\ 0 \cdots 0\ b]$.

Pivots in columns 1,3,4 so x_2 is free.

Read off equations from R.E.F. matrix:

$$\begin{aligned} x_1 &= 5 - 3x_2 \\ x_2 &= \text{free} \\ x_3 &= 1 \\ x_4 &= -2 \end{aligned}$$

2. (a) The matrix $\begin{bmatrix} 1 & 0 & 3 & 0 \\ \underline{1} & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix}$ in neither in echelon form nor reduced echelon form. The leading entry must move to the right as go down the rows. One problem entry is underlined.

(b) False: It is true that *if consistent*, whenever a system has free variables, the solution set contains many solutions. But free variables do *not* require consistency. This was a hard logic one to realise, and very few of you did.

(c) It is possible to multiply a 5-by-3 matrix by a length 3 vector. Remember matrix is 5 rows by 3 columns, and a 'vector' means a column vector (this is how we introduced vectors).

$$A\mathbf{x} = \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} * \\ * \\ * \end{bmatrix} = \begin{bmatrix} * \\ * \\ * \\ * \\ * \end{bmatrix}$$

A length-5 (column) vector results, *i.e.* a vector in $\mathbb{R}^5$. See p.41 Example 1, and p.47 Practise Problem 2. In general, if A is $m \times n$, then $\mathbf{x}$ must be in $\mathbb{R}^n$, and $A\mathbf{x}$ is in $\mathbb{R}^m$.

3. Call the first vector $\mathbf{b}$ and the 3 vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$. Try to solve the linear system $[\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3\ \mathbf{b}]$.

$$\begin{bmatrix} 1 & 0 & 2 & -1 \\ 3 & -1 & 5 & -4 \\ 5 & 2 & 12 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & -1 \\ & 1 & 1 & 1 \\ & & & -6 \end{bmatrix}$$

which is inconsistent, so $\mathbf{b}$ cannot be written as a linear combination of the 3 vectors.

The 3 latter vectors do not span $\mathbb{R}^3$, because we have found a vector $\mathbf{b}$ in $\mathbb{R}^3$ which *cannot* be written as a linear combination of them. That is, $\mathbf{b}$ is not in $\mathrm{Span}(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$, so the Span cannot be all of $\mathbb{R}^3$.

Another way to see this is to look at the echelon form of the coefficient matrix (all but the last column of the echelon form of the augmented matrix):

$$\begin{bmatrix} \underline{1} & 0 & 2 \\ 0 & \underline{1} & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

which has only 2 pivots (underlined). It would require a pivot in every row to span the space ($\mathbb{R}^m$ with $m = 3$) in which the column vectors lie.