V63.0140-3: Linear Algebra. MIDTERM

Tues 10/16/03. 60 minutes, 60 points. Please answer on this sheet, write your name at the top. Label your answer, show your working, and explain wherever you can. Good luck!

1. [9 points] Solve the linear system $A\mathbf{x} = \mathbf{b}$ given

$$A = \begin{bmatrix} 2 & -6 & 1 & -2 \\ -1 & 3 & 2 & 6 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

(a) If inconsistent, explain why. If consistent, write the general solution in parametric *vector* form (*i.e.* in the form $\mathbf{x} = \mathbf{p} + s\mathbf{u} \cdots$ etc):

$$\mathbf{x} =$$

(b) Write in the same form the solution to the corresponding homogeneous problem $A\mathbf{x} = \mathbf{0}$:

$$\mathbf{x} =$$

2. [10 points]

(a) Define the concept of linear independence.

(b) Fixing $h = 0$, Is the set of three vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ h \end{bmatrix}$, linearly independent?

(c) Still fixing $h = 0$, is the last of these vectors in the span of the first two vectors?

(d) What *condition* on h is required if the set of three vectors is to span $\mathbb{R}^3$?

3. [6 points] A linear transformation T can be written as a function $T(x_1, x_2) = (3x_2,\ x_1 - x_2,\ 2x_1)$.

(a) Find the *standard matrix* for this transformation (check you have the correct size matrix):

$$A =$$

(b) Is T onto $\mathbb{R}^3$? (why?)

4. [8 points]

(a) A matrix A has been factored into $A = LU$ where

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}, \qquad U = \begin{bmatrix} 2 & 0 & 1 \\ 0 & -3 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

Use this factorization to solve $A\mathbf{x} = \mathbf{b}$, for the case $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$:

(b) Is the matrix A invertible? (why?)

5. [7 points]

   (a) True/false: if a system of linear equations has two different solutions, then it must have infinitely many solutions.

   (b) True/false: a set of three vectors in $\mathbb{R}^2$ can be linearly independent. (Explain your answer)

   (c) A transformation T from $\mathbb{R}^2$ to $\mathbb{R}^2$ maps $(1, 0)$ to $(3, 4)$ and $(2, 0)$ to $(6, 7)$. What (if anything) can we say about whether T is a *linear* transformation?

6. [9 points] Consider a (dubious) economy with two sectors, pizza (P) and beer (B). Let x_1 be the units of output (*i.e.* dollars of production) by the pizza sector, and x_2 be the units of output by the beer sector. Recall that Leontief's equation is $\mathbf{x} = C\mathbf{x} + \mathbf{d}$. Suppose the consumption matrix is $C = \begin{bmatrix} 2/5 & 1/5 \\ 2/5 & 3/5 \end{bmatrix}$.

   (a) How much *intermediate demand* for beer is created by each unit production of pizza? (Hint: you do not need to solve anything!)

   (b) What output vector $\mathbf{x}$ satisfies a *final demand* $\mathbf{d}$ of zero units of pizza and 4 units of beer?

7. [11 points]

   (a) Find the determinant of the following matrix, taking care to get the correct sign. You may use any method [hint: one is much easier than the other]. State which you use.

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 4 \\ 2 & 3 & 3 \end{bmatrix}$$

(b) Use your above result to find the determinant of this matrix:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 7 & 1 & 1 \\ 2 & -9 & 2 & 4 \\ 2 & 5 & 3 & 3 \end{bmatrix}$$

(c) Prove that for $n \times n$ matrices A and B, where A is invertible, $\det(A^{-1}BA) = \det B$.

(d) True/false: If a $n \times n$ matrix has determinant zero, its rows can span $\mathbb{R}^n$?