V63.0140-3: Linear Algebra. FINAL

Thurs 12/18/03. 110 minutes, 90 points. Please answer on these sheets, write your name at the top. Tackle the questions in any order, show your working, explain wherever you can. Good luck & happy holidays!

1. [17 points] A set of vectors in $\mathbb{R}^3$ is given by $W = \left\{ \begin{bmatrix} a+2b+2c \\ -2b+c \\ a+3c \end{bmatrix} a, b, c \text{ real} \right\}$.

   (a) [4 points] Determine, using the tests for a subspace, whether W is a subspace of $\mathbb{R}^3$. (Explain any claims you make).

   (b) [4 points] Find a basis for W.

(c) [2 points] What is dim W?

(d) [3 points] Give a definition of the set $W^{\perp}$.

(e) [4 points] Find a basis for $W^{\perp}$.

2. [14 points] Consider the symmetric matrix $A = \left\{ \begin{array}{cc} 2 & 5 \\ 5 & 2 \end{array} \right\}$, which has eigenvalues $\lambda = -3, 7$.

   (a) [4 points] Find the eigenvectors.

(b) [4 points] Give a formula for $\mathbf{x}_k$, the k^{th} iterate of the discrete dynamical system $\mathbf{x}_k = A\mathbf{x}_{k-1}$, with initial vector $\mathbf{x}_0 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$. Your formula should involve only numbers and k.

(c) [1 point] To which direction will $\mathbf{x}_k$ tend in the limit $k \to \infty$?

(d) [2 points] Classify the quadratic form $Q(\mathbf{x}) = 2x_1^2 + 10x_1x_2 + 2x_2^2$ (is it positive/negative definite, or indefinite?)

(e) [3 points] If P is the matrix of normalized eigenvectors of A stacked in columns, and we change variable to $\mathbf{y} = P^T\mathbf{x}$, express the above quadratic form Q as a function of y_1 and y_2, the components of $\mathbf{y}$.

3. [12 points]

(a) [2 points] Do the vectors $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 0 \\ 15 \\ -2 \end{bmatrix}$ form an orthogonal set?

(b) [6 points] Construct an ortho*normal* basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ for $\text{Span}\{\mathbf{x}_1, \mathbf{x}_2\}$, with $\mathbf{x}_1$, $\mathbf{x}_2$ as given above. [Hint: one way round is much easier than the other!]

(c) [4 points] Now consider a general square $n \times n$ matrix A whose columns form an orthonormal set. Prove the remarkable result that the *rows* of A also form an orthonormal set. [Hint: consider the product $A^T A$].

4. [16 points] In parts a–c you don't necessarily need to diagonalize the matrix in order to answer the question.

(a) [3 points] Is the matrix $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 4 \end{bmatrix}$ diagonalizable? Why?

(b) [4 points] Is the matrix $B = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{bmatrix}$ diagonalizable?

(c) [4 points] Is the matrix $C = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{bmatrix}$ diagonalizable?

(d) [5 points] Find a matrix P and a diagonal matrix D such that the above matrix can be written $C = PDP^{-1}$.

$$P = \begin{bmatrix} \quad & \quad & \quad \\ \\ \\ \end{bmatrix} \qquad D = \begin{bmatrix} \quad & \quad & \quad \\ \\ \\ \end{bmatrix}$$

5. [10 points] You are given $A = \begin{bmatrix} 1 & -2 \\ -2 & 4 \\ 1 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 5 \\ 0 \\ 4 \end{bmatrix}$.

(a) [6 points] Find $\hat{\mathbf{x}}$, the least squares solution of the inconsistent system $A\mathbf{x} = \mathbf{b}$. [Hint: make sure you notice a common factor which simplifies the arithmetic, otherwise re-check your work!]

(b) [4 points] What is the least squares error associated with your solution?

6. [10 points]

(a) [5 points] $\mathcal{B} = \{1, 1+2t, -1+t^2\}$ is a basis for $\mathbb{P}_2$, the vector space of all degree-2 polynomials. Find the $\mathcal{B}$-coordinate vector $[\mathbf{p}]_{\mathcal{B}}$ of the polynomial $\mathbf{p}(t) = 1 + 2t + 3t^2$.

(b) [5 points] $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ is a basis for $\mathbb{R}^2$, with $\mathbf{c}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{c}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Also $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ is another basis for $\mathbb{R}^2$ defined by $\mathbf{b}_1 = 2\mathbf{c}_1 - 2\mathbf{c}_2$ and $\mathbf{b}_2 = \mathbf{c}_1 + 2\mathbf{c}_2$. Find the change of coordinates matrix $P_{\mathcal{C}\leftarrow\mathcal{B}}$ such that for any vector $\mathbf{x}$ we have $[\mathbf{x}]_{\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}\,[\mathbf{x}]_{\mathcal{B}}$.

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

7. [11 points]

(a) [1 point] True/false: two eigenvectors belonging to different eigenvalues are always linearly independent?

(b) [1 point] True/false: two eigenvectors belonging to the same eigenvalue are always linearly dependent?

(c) [1 point] True/false: The steady-state (long-time limit) vector of a Markov chain with stochastic matrix A is an eigenvector of A with eigenvalue 1?

(d) [1 point] True/false: A symmetric matrix is always diagonalizable even if its eigenvalues are not distinct?

(e) [2 points] A real 3×3 matrix has eigenvalues 2 and $-4 - i$. Are there any more eigenvalues, if so of what value?

(f) [2 points] What is the rank of a 5×3 matrix if a basis for its null space contains only 1 vector?

(g) [1 point] What is the largest possible dimension of the row space of a matrix of size 7×4?

(h) [2 points] True/false: The product of 2 symmetric matrices is always itself symmetric? [Hint: try examples if stuck].