

## V63.0123-1 : Calculus III. Sample Midterm2 Answers

1. [12 points]

Let  $f(x, y) = \frac{x}{y} + \frac{y}{x}$ .

(a)  $D = \{(x, y) | x \neq 0, y \neq 0\}$

(b)  $\nabla f(x, y) = \left(\frac{1}{y} - \frac{y}{x^2}, \frac{1}{x} - \frac{x}{y^2}\right)$ .

(c)  $\nabla f \cdot \mathbf{r} = 0$  for all  $(x, y)$ , by using  $\mathbf{r} = (x, y)$ . In other words  $\left(\frac{1}{y} - \frac{y}{x^2}\right)x + \left(\frac{1}{x} - \frac{x}{y^2}\right)y$  simplifies to zero. Lengths are not zero  $\Rightarrow$  perpendicular.

(d) No. If approach along  $y = x$ , lim is  $1 + 1 = 2$ . If approach along  $x$ -axis or  $y$ -axis, lim is infinite.

2. [10 points]

Treat as Type 1: for given  $(x, y)$ , limits on  $z$  integral are 0 and  $1 - x$ . Remaining shadow region in  $xy$  plane has bounds  $x = y^2$  and  $x = 1$ , treat as Type II is easiest. Ans:

$$\int_{-1}^1 \int_{y^2}^1 \int_0^{1-x} x \, dz \, dx \, dy = 1/3 - 1/5 + 2/21 = 8/35. \quad (1)$$

3. [8 points]

Domain is semicircle in 1st and 4th quadrants. Assume density, our function  $f(x, y) = 1$ . Then  $m = \text{Area} = \text{half that of circle} = \pi/2$ . Also  $x = r \cos \theta$ .

$$\bar{x} = \frac{1}{m} \int_{-\pi/2}^{\pi/2} \int_0^1 r \cos \theta \, r \, dr \, d\theta = \frac{4}{3\pi}. \quad (2)$$

4. [10 points]

Tree has  $f$  at top,  $x, y, z$  in middle,  $s, t$  as each sub-branch (apart from  $z$  which only has sub-branch  $t$ ).

$$\frac{\partial f}{\partial s} = (y + z)t + (x + z)te^{st} \rightarrow 3 \quad \text{at } (s, t) = (0, 1)$$

$$\frac{\partial f}{\partial t} = (y + z)s + (x + z)se^{st} + 2(x + y)t \rightarrow 2 \quad \text{at } (s, t) = (0, 1)$$

5. [10 points]

$$f_x = 2xy = 0 \text{ so } x = 0 \text{ or } y = 0.$$

$$f_y = x^2 + 2y - 1 = 0. \text{ Use } x = 0 \text{ gives } y = 1/2. \text{ Use } y = 0 \text{ gives } x = \pm 1.$$

$$(0, 1/2) : D = f_{xx}f_{yy} - f_{xy}^2 = 2(2y) - 4x^2 = +2. \text{ Minimum.}$$

$$(1, 0) \text{ and } (-1, 0) : D = -4. \text{ Saddles.}$$

$$\text{Contour line } f = 0 \text{ gives } y(x^2 + y - 1) = 0 \text{ that is } y = 0 \text{ or } y = 1 - x^2.$$

