V63.0123-1 : Calculus III. Derivative of products

from lecture Wed Feb 5

Suppose you have a scalar function $f(t)$ multiplying a vector function $\mathbf{a}(t)$. What is the derivative of this product with respect to t?

I use abbreviation prime to mean derivative with respect to t, and omit the dependence on t sometimes for clarity.

$$\begin{aligned}
\frac{d}{dt}[f(t)\mathbf{a}(t)] &\overset{\text{derivative rule for components}}{=} \left(\frac{d}{dt}(fa_1),\ \frac{d}{dt}(fa_2),\ \frac{d}{dt}(fa_3)\right) \\
&\overset{\text{product rule on each component}}{=} (f'a_1 + fa_1',\ f'a_2 + fa_2',\ f'a_3 + fa_3') \\
&\overset{\text{split into sum of 2 vectors}}{=} (f'a_1,\ f'a_2,\ f'a_3) \ +\ (fa_1',\ fa_2',\ fa_3') \\
&\overset{\text{recognize each as scalar multiple of a vector}}{=} f'(t)\,(a_1, a_2, a_3) \ +\ f(t)(a_1', a_2', a_3') \\
&\overset{\text{recombine components using derivative rule}}{=} f'(t)\,\mathbf{a}(t) \ +\ f(t)\mathbf{a}'(t). \qquad (1)
\end{aligned}$$

So the product rule of differentiation works as you might expect, for a scalar-vector product.

Also see the Stewart book p. 880, where the dot product is done.

It also works for cross product:

$$(\mathbf{a} \times \mathbf{b})' \ =\ \mathbf{a}' \times \mathbf{b} \ +\ \mathbf{a} \times \mathbf{b}' \qquad (2)$$

Note the order of $\mathbf{a}$ coming before $\mathbf{b}$ doesn't change! The cross product is not commutative!