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Abstract
This paper presents one of the techniques for solving inverse source problems involving waves. We studied the effects of frequency of waves on resolving images obtained via detectors. However, inverse source problems present a challenge in medical imaging. The difficulty is ill-posedness: the answer is arbitrarily sensitive to measurement data. In this study, we presented a step-by-step way of solving inverse source problems using Singular Value Decomposition (SVD). The goal was to select the appropriate regularization factor and thus constrain the solution to realistic values. This is a common method used in breast cancer detection by microwave tomography. We concluded that the regularization factor is dependent on the noise level that arises in measuring the data.

1. Introduction

Medical imaging is a field which is receiving increasing attention as medicine strives for ways to non-invasively image the human body for diagnosis and functional study. Several popular imaging technologies require the solution of an "inverse problem" in order to obtain an image. Examples of such medical imaging techniques include electrical impedance tomography, diffuse optical tomography, X-ray tomography, microwave scattering, reflection tomography, and inverse source problems like electroencephalogram (EEG)(1,2,3,4).

An inverse problem involves using results of an observation to determine the parameters of a system. It is therefore the reverse of the "forward problem" which involves predicting observations given the parameters of the system. Inverse problems are, thus, much more complicated than forward problems. A common setback with inverse problems is the concept of ill-posedness. A problem is defined as ill-posed if its answer is arbitrarily sensitive to measurement data. In the real world, measurements are limited to certain accuracies. Noise always perturbs the data, causing changes in the solution. With an ill-posed problem, small changes in data can lead to arbitrarily large changes in solutions which may overwhelm the desired image.

In this introductory paper, we discuss the mathematics of solving a type of inverse problem known as inverse source problems: we seek to identify amplitudes of the sources of waves and thereby construct an image, given only the total fields measured at a set of detectors. The sources are supposedly inside the body, and the detectors are necessarily outside for non-invasive imaging. We study a simple two-dimensional single frequency Helmholtz model problem that illustrates the essential practical issues arising in more complicated real-world imaging problems.

Given a set of known detector locations and unknowns (source locations and strengths), the goal is to detect the internal properties of the sources via the amplitudes at the detectors. Some of the problems of interest include whether we can distinguish nearby sources from one another.
One major way to approach this problem of ill-posedness is via "regularization." By regularizing, we constrain the solution to realistic values, thus maintaining image quality even in the presence of noise.

2. Mathematical statement of problem

We consider n sources in the plane at locations z_i, for $i=1... n$, and detectors at y_j, $j = 1...m$, (See Figure 1).

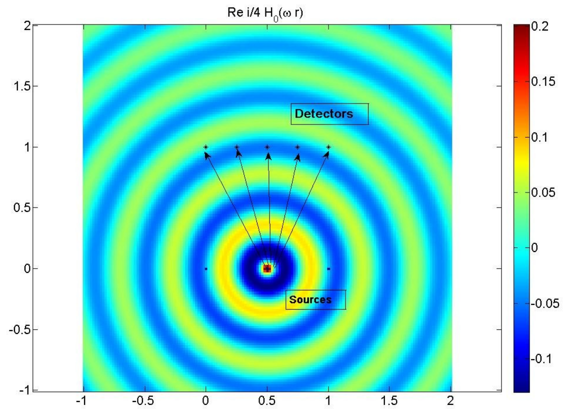

Figure 1 – Single source on the x-axis and five detectors on the line y=1 with each detector detecting the field from the source.

Scalar waves at fixed frequency, ω, satisfy the Helmholtz equation. The fundamental solution which is the field due to a point source, involves the Hankel function which only depends on the distance from sources,

$$g(z) = \frac{i}{4} H_0^{(1)}(\omega|z|) \quad \textbf{(1)}$$

Thus the field at a point y_i due to source at z_j is:

$$g(z_j - y_i) = \frac{i}{4} H_0^{(1)}(\omega|z_j - y_i|) \quad \textbf{(2)}$$

The total field at y_i is the weighted sum of the above, i.e.: $\sum_{j=1}^{n} x_j g(z_j - y_i)$,

where x_j are the source amplitudes.
We stack the measured fields at detectors onto a column vector ***b***

And thus $b_i = \sum_{j=1}^{n} A_{ij} x_j$ **(3)**

When we consider the imaging system as a linear inverse problem, the linear system is defined by: $Ax = b$, **(4)**
where A is the influence matrix or measurement process, b is the measurements vector and x is the unknown strengths or intensities.
Due to properties which form the "image" of matrix A, it is possible that there is no value of x which solves this equation. However, we seek the best solution in the least square sense.
In summary, the goal of solving the equation in the least square sense is to find the set of vectors of x which minimizes the distance $|Ax - b|$, such that, Ax in the column space of A is closest to b.

<u>Proposition 1</u>: Let A be an m by n matrix, and $b \in R^m$ a vector. Any x which minimizes the above least-squares problem is a solution to the "normal equation" $A^T Ax = A^T b$.

PROOF:

Writing $J(x) = |Ax - b|^2$, we have that $\frac{\partial}{\partial x_i}|Ax - b|^2 = 0$, for *i=1…n,* for *x* at its minimum (whether unique or not). Our task is now to show that $\frac{\partial J}{\partial x_i} = -(2b^T A)_i + 2(A^T Ax)_i$

We write $\min_x J(x) = \min_x |Ax - b|^2$ **(5)**

$$= \min_x [(Ax - b).(Ax - b)]$$

$$= \min_x ((Ax).Ax - (Ax).b - b.Ax + b.b)$$

We tale derivatives of each term

$\frac{\partial b^T Ax}{\partial x_i} = \frac{\partial}{\partial x_i}\sum_{jk} b_j^T A_{jk} x_k = \sum_{jk} b_j^T A_{jk}\delta_{ki} = (b^T A)_i$, (using the property of Kronecker delta, δ_{ki})

Also, $\frac{\partial (x^T A^T b)}{\partial x_i} = \frac{\partial}{\partial x_i}\sum_{jk} x_k A_{kj} b_j = \sum_{jk} \delta_{ki} A_{kj} b_j = (b^T A)_i$

$$\frac{\partial (x^T A^T Ax)}{\partial x_i} = \frac{\partial}{\partial x_i}\sum_j (\sum_k A_{jk} x_k)^2 = \sum_{jk} 2(A_{jk} x_k).\frac{\partial}{\partial x_i}\sum_k A_{jk} x_k$$

$$\sum_j 2(\sum_k A_{jk} x_k).A_{ji} = 2\sum_{jk} A_{ij}^T A_{jk} x_k = 2(A^T Ax)_i$$

Combining the elements,

$\frac{\partial J}{\partial x_i} = -(2b^T A)_i + 2(A^T Ax)_i$ for each *i*

$0 = -A^T b + A^T Ax$, which is the normal equation. QED. **(6)**

3. Linear Algebra tools to handle ill- posedness

a) Singular Value Decomposition

We often resort to SVD of a matrix as a tool to solve linear inverse problems. SVD is the factorization of a matrix into a product of three matrices that reveal the structure of the original matrix. An *m* by *n* matrix, *A*, can be factorized such that:

$$A = U\Sigma V^T, \quad \textbf{(7)}$$

where V^T is the transpose of *V,* and Σ an *m* by *n* diagonal matrix with the diagonal entries σ_j positive and non-increasing.

U is an *m* by *m* unitary matrix, such that $UU^T = I$, and V is a unitary *n* by *n* matrix, such that $VV^T = I$.

SVD is motivated by the geometric fact that a hyperellipse is the image of a unit sphere under transformation by any matrices *(5).* The hyperellipse is the surface obtained when a unit sphere is stretched by some factors in some directions.

b) Regularization

Regularization involves introducing a parameter to impose stability when solving an ill-posed problem. "Tikhonov regularization" is the most familiar method used. A regularization term is added in the minimization of the distance $|Ax - b|$ to obtain:

$$J(x) = |Ax - b|^2 + \alpha^2 |x|^2$$

where $\alpha > 0$ is the regularization parameter. **(6)**

In similar fashion to proposition 1, minima $\hat{x}$ of J satisfy:

$\hat{x} = (A^T A + \alpha^2 I)^{-1}.A^T b$

We now show that $\hat{x}$ is unique and can be found via SVD.

For, $(A^T A + \alpha^2 I).\hat{x} = A^T b$

$= (V \sum^T U^T U \sum V^T + \alpha^2 I)^{-1} (V \sum^T U^T) b$

$= (V \sum^T U^T U \sum V^T + \alpha^2 V I V^T)^{-1} (V \sum^T U^T) b$

$= V(\sum^T \sum + \alpha^2 I)^{-1} \sum^T U^T b$

$= V diag\left\{ \frac{\sigma_j^2}{\sigma_j^2 + \alpha^2} . \frac{1}{\sigma_j} \right\} U^T b$

This formula for the reconstructed image also shows that for $\alpha > 0$, $\hat{x}$ is unique but

The matrix $diag\left\{ \frac{\sigma_j^2}{\sigma_j^2 + \alpha^2} . \frac{1}{\sigma_j} \right\}$ has the effect of inverting the σ_j's above α, killing the smaller ones. Thus regularization "cuts off" the singular values that are less than α.

6. Results and Data analysis

We now apply the above to a simple layout of sources and detectors: a line of sources physically separated from a line of detectors (since one cannot probe directly inside the body).

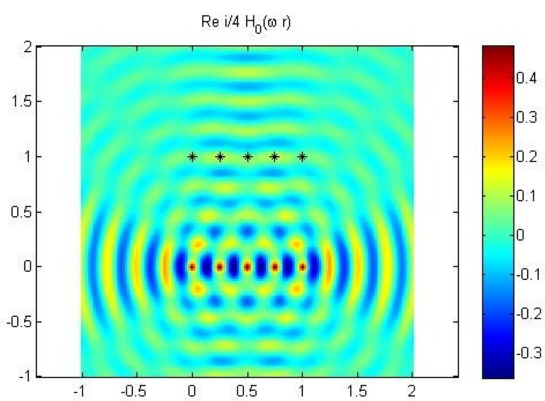

Figure 2: Five sources and five detectors with waves at a frequency of ω=24 and equal amplitudes.

When reconstruction is done using regularization, one must choose a regularization parameter, α, which removes the background noise but does not cause too much blurring, i.e. loss of resolution due to cutting off the effect of singular values lying below α.

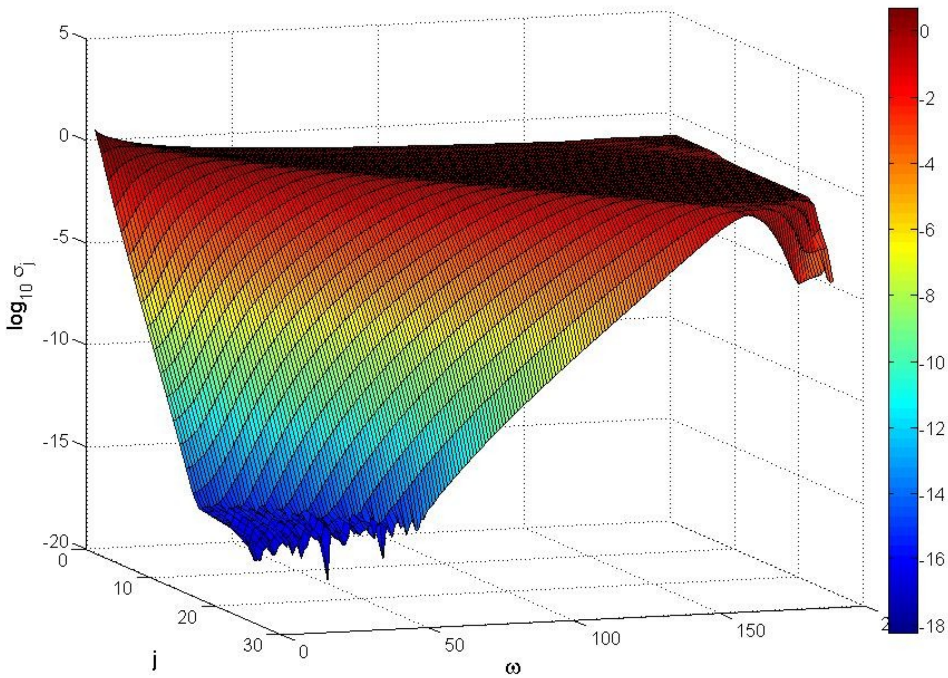

Figure 3: 3D representation of set of singular values vs frequency ω for 25 sources and 25 detectors. Note that the number of singular values above a given α increases linearly with frequency ω.

As already discussed, singular values, σ_j's are positive and non-increasing in the diagonal matrix $\sum$ of the SVD; figure 3 shows the decay of the singular values.

We observed in this study that each number of sources and detectors had a unique frequency level, "the ideal frequency", at which reconstruction occurred with minimum noise or blurring – the position with the highest bump in Figure 3 (where ω is around 150). Since the frequency cannot necessarily be chosen at will in the real world, focus must be placed on finding an appropriate regularization parameter for other, usually smaller, ω.

In setting a good regularization parameter, our goal is to remove very small singular values which will blow up and create too much noise if not removed when the inverses are found. However, we also do not want to remove too many singular values by choosing a large α, as the image can then not be resolved.

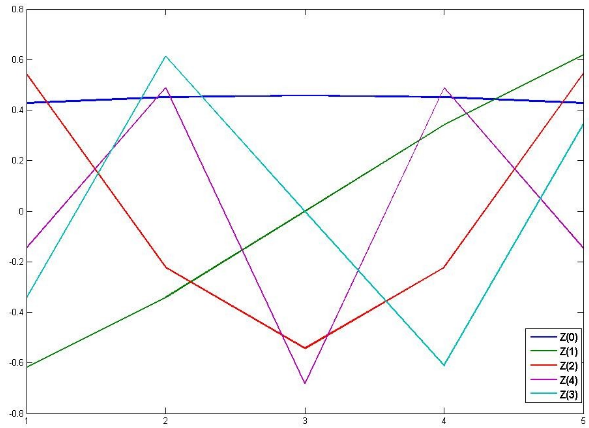

Figure 4: Singular vectors exhibiting fourier modes characteristics. Oscillations are more rapid for larger index j where σ_j are smaller.

The research showed that the singular vector i.e., the columns of V, exhibit characteristics of Fourier modes. Thus, choosing a large α makes resolution worse by excluding δ_j's and singular vectors with high fourier oscillations necessary for good resolution. (See Figure 4)

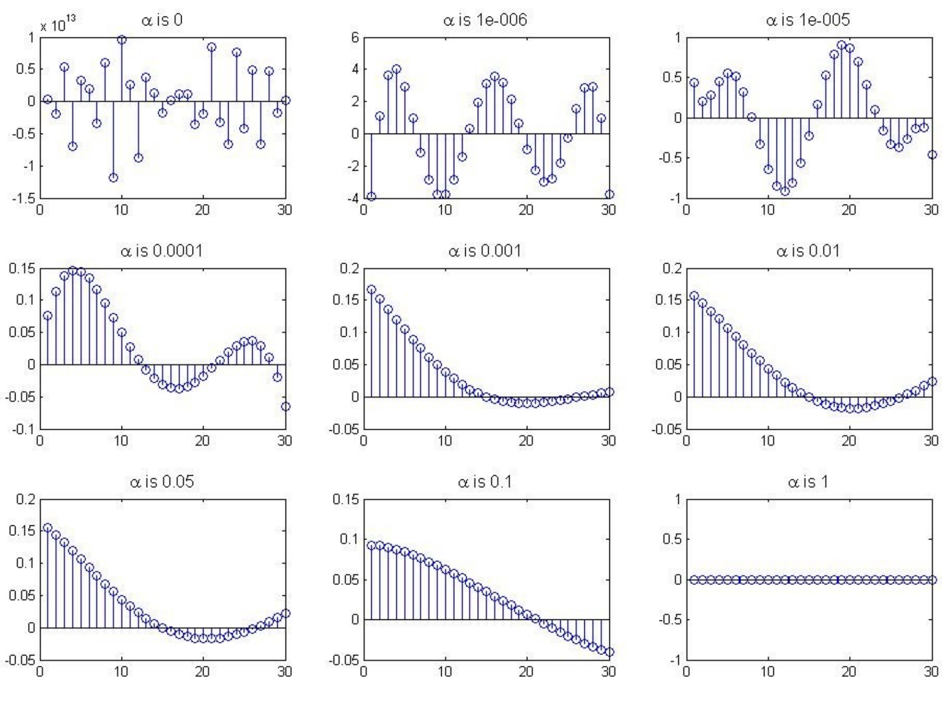

Figure 5: Stem plots of reconstructed 1D images at a low frequency, ω=5, and addition of noise at 10^{-2}. The original image with no noise nor blurring should be a single stem plot at position 4 on the x-axis. As the regularization parameter α increases, resolution gets worse but noise gets better. $10^{-4} \leq \alpha \leq 10^{-3}$ produces images closest to the original stem plot.

We may plot an image as a function of position along some line, i.e., the set of source amplitudes x ;. When various reconstruction graphs are made with different regularization parameters α, as seen in Figure 5, we are able to choose by eye the appropriate regularization parameter that removes enough noise but does not cause too much blurring. However, we may also rigorously choose this level, see Figure 6. In Figure 5, we chose a low frequency that makes high-resolution reconstruction impossible.

7. Conclusion

Finding unknown source amplitudes from detected waves is an important problem in medical imaging and linear algebra enables us to solve this.
There is a trade-off between eliminating noise and avoiding image blurring in a reconstructed image. Increasing the regularization parameter will remove noise since the effects of small singular values in the SVD are removed. However, too much of this will prevent good reconstruction due to blurring. The goal is to select the best regularization parameter; other studies have shown that the noise levels set the best choice for the regularization parameter*(6)*.

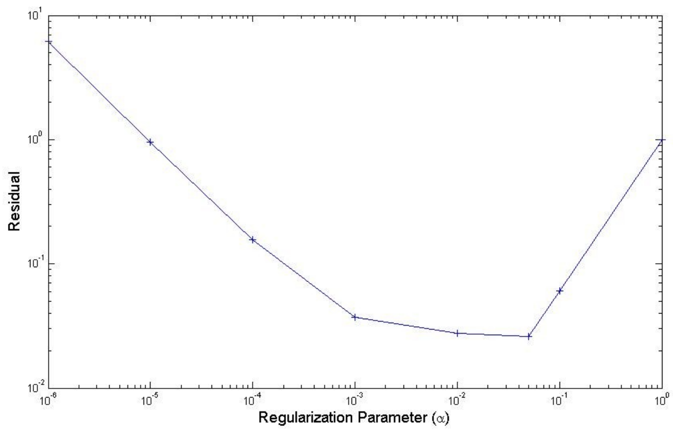

Figure 6: - Graph of residual (difference between original image and regularized image) against various regularization parameters. The lower the residual, the more accurate the answer.

From this research, we observed that the lowest residual (that is the difference between the original answer and the regularized answers) occurs when the regularization parameter is approximately equal to the noise level from the various noise levels used (Figure 6). Finding ways to estimate noise levels will improve results in the inverse source problem and could have great benefits in the field of medical imaging.

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