Asymptotic Growth of Associated Primes of Graph Ideals

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Introduction

We specify a class of graphs, H_t, and characterize the irreducible decomposition of all powers of the cover ideals. This gives insight into the structure and stabilization of the corresponding associated primes; specifically, providing an answer to the question "For each integer $t \geq 0$, does there exist a (hyper)graph H_t such that stabilization of associated primes occurs at $s \geq (\chi(H_t) - 1) + t$?" [4]. For each t, H_t has chromatic number 3 and associated primes that stabilize at $s = 2 + t$.

1. Motivation

We study the interaction between graph theory and algebra through the association of ideals to graphs. On the algebraic side, studying powers of ideals, especially high powers, comes from a classical problem of algebraic geometry — that of resolving singularities. One can associate a ring, R, to a singular curve and an ideal, I, to the singularity, then use a blowup ring of the form $R \oplus I \oplus I^2 \oplus I^3 \oplus ...$ as a first step in resolving the singularity. This is one motivation for studying powers of ideals and in turn the asymptotic growth of associated primes of these ideal powers.

Brodmann [1] in 1979 proved that the set of associated primes of powers of ideals is finite and thus that it *stabilizes* for high powers of ideals. In other words, there exists some positive integer s such that $Ass(R/I^s) = Ass(R/I^n)$ for all $n \geq s$. Thus, the primary decomposition of an ideal and the corresponding associated primes give a framework in which to study the ideal because if we find the stabilization point for the associated primes, we get a sense of 'sameness' for the powers of ideals past that point. Brodmann's result leads to natural questions about the stabilization value, s, and how to bound it.

Some answers lie at the intersection of graph theory and algebra. Recently, Francisco, Ha, and Van Tuyl [3],[4], related asymptotic prime stability of certain graph ideals to the *chromatic number* of the graphs themselves. The chromatic number, $\chi(G)$, of a graph G is not only of interest theoretically, but can provide solutions to efficiency problems. For example, if we represent signal towers with the vertices of a graph we can connect them by edges if their ranges overlap. The chromatic number, or minimal coloring so that no two vertices are adjacent, gives the minimal number of channels needed to broadcast without interference between towers with overlapping ranges. This efficiency problem [5] and others like it often appear in standard graph theory textbooks like [2].

2. The Question

Francisco, Ha, and Van Tuyl give a bound on the asymptotic growth of associated prime ideals for powers of cover ideals, proving that the primes stabilize for some $s \geq \chi(G) - 1$ [4]. They show this bound is not sharp, leading to the question: "For each integer $t \geq 0$, does there exist a (hyper)graph H_t such that stabilization of associated primes occurs at $s \geq (\chi(H_t) - 1) + t$?" [4][Question 4.9].

3. Graph Theory and Associated Primes

We briefly introduce the important structures from graph theory and algebra needed for our results.

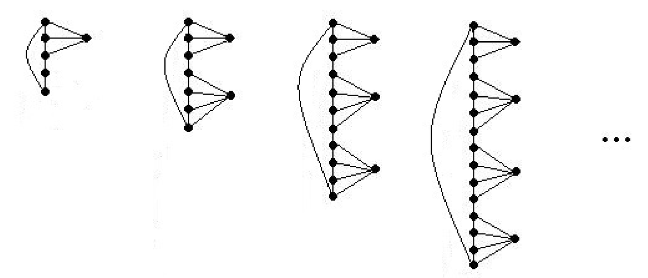
First, we define a graph G by a vertex set, $V(G)$, and edge set, $E(G)$. As we work with simple graphs, the subsets in the edge set will have order 2.

We assign a variable in a polynomial ring to each vertex of a graph. This assignment allows us to associate several different monomial ideals to the graph. The *edge ideal*, $I(G)$ is the ideal formed by taking all $x_i x_j$ as generators if $\{x_i, x_j\} \in E(G)$. We take the *Alexander dual* of I by mapping these generators to primary components. This ideal is also referred to as the *cover ideal* because the generators correspond to vertex covers of the graph [4].

Every primary ideal has an associated prime ideal, $\sqrt{P}$, and the set of these associated prime ideals of the primary decomposition of an ideal I forms the *associated prime ideals* of I. As the irreducible decomposition of monomial ideals is easier to work with than the primary decomposition in this context, we find *irreducible decompositions*. The set of associated prime ideals corresponding to the irreducible decomposition of an ideal are the same as the set of associated prime ideals corresponding to the primary decomposition [6]; thus, to compute $Ass(R/J^n)$ we will find the associated prime ideals of the irreducible decomposition of J^n.

4. A Class of Graphs

We proceed to define a class of graphs, H_t, for which the stabilization of powers of the cover ideal is arbitrarily large. We form the graphs by strategically adding vertices to an odd cycle of length $(4t - 1)$ for $t > 1$.



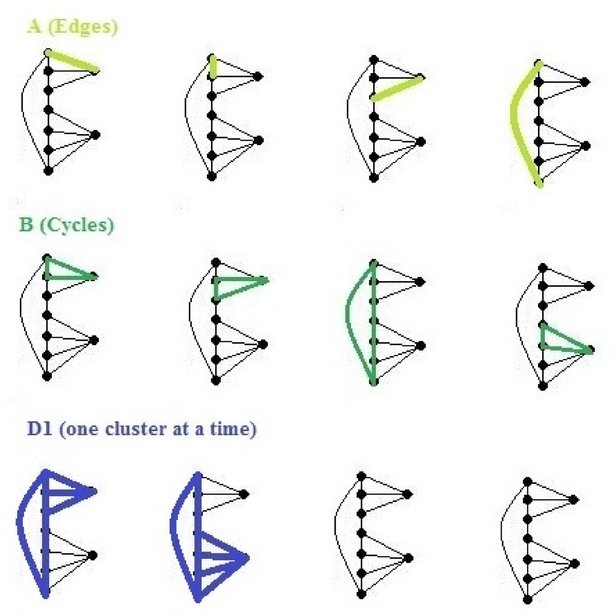
We note that H_t has chromatic number three for all $0 < t < \infty$ and thus we need show that the set of associated primes for the cover ideals of H_t stabilizes at $s = t + 2$. This theorem follows from a characterization of the irreducible decomposition of all powers of the cover ideals, J^n, for these graphs, for which we use the structure of the graphs and the fact that N is a monomial element of J^n if and only if the variable powers of N are an n-cover of G that can be written as the sum of n one-covers of G.

5. The Main Result: Decomposing Cover Ideals

Theorem 1. *Let $G = H_t$ for $i \in \mathbb{N}$. Let J be the cover ideal of G so $J = I(G)^\vee$. Then J^n has the following irreducible decomposition:*

$$J^n = A \cap B \cap \bigcap_{r=1}^{n-2} D_r.$$

This theorem describes when sets of vertices appear in powers of the cover ideal, and what degree the vertices have. The sets of vertices are important to answering Francisco, Ha, and Van Tuyl's question, and can be seen visually as:



6. Conclusion

The answer to the question posed now follows as a corollary.

Theorem 2. *For all $n \in \mathbb{N}$, the associated primes of H_t stabilize at $s = 2 + t = (\chi(H_t) - 1) + t$.*

In J^n, $n - 2$ y-vertices, to some power, are generators of components of the irreducible decomposition. The maximal ideal for a graph with t y-vertices thus is a component of the irreducible decomposition of J^{t+2}. As associated prime ideals follow from the irreducible decomposition, the associated primes for this graph can not stabilize until J^{t+2}. In fact, they must stabilize at J^{t+2} because once the maximal ideal appears, no new ideals appear as components of the irreducible decomposition.

7. Further Directions

First we notice that the class of graphs, H_t, is doubly infinite. The proofs remain unchanged for any odd cycles greater than $4t - 1$ for $t > 1$ and for any odd cycle greater than 5 for $t = 1$ as long as they have the same y attachments.

Similarly, the $\{y_i \cup N(y_i)\}$ segments need not always consist of 5 vertices for $i \neq 1$. For example, for certain graphs, cover ideals of graphs with y-segments of $2k + 1$ vertices follow the same pattern as in Theorem 1.

Finally, we can interpret this theorem from the perspective of determining chromatic number. Adding edges to H_t that do not affect the proof of Theorem 1 gives graphs with associated prime stabilization of $t+2$. We can then interpret the theorem as $\chi(H_t) = s - t + 1 = 3$ to find the chromatic number of these graphs. For example, adding an edge between each successive y_i does not affect the proof of Theorem 1. Thus, these graphs have the same stabilization values, and therefore still have chromatic number 3.

References

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