Difference Equations
to
Differential Equations

Section 4.8

Distance, Position, and the Length of Curves

Although we motivated the definition of the definite integral with the notion of area, there are many applications of integration to problems unrelated to the computation of area. Depending on the context, the definite integral of a function f from a to b could represent the total mass of a wire, the total electric charge on such a wire, or the probability that a light bulb will fail sometime in the time interval from a to b. In this section we will consider three applications of definite integrals: finding the distance traveled by an object over an interval of time if we are given its velocity as a function of time, finding the position of an object at any time if we are given its initial position and its velocity as a function of time, and finding the length of a curve.

Distance

Suppose the function v is continuous on the interval $[a, b]$ and, for any $a \leq t \leq b$, $v(t)$ represents the velocity at time t of an object traveling along a line. Divide $[a, b]$ into n time intervals of equal length

$$\Delta t = \frac{b-a}{n}$$

with endpoints $a = t_0 < t_1 < t_2 < \cdots < t_n = b$. Then, for $j = 1, 2, 3, \ldots, n$, $|v(t_{j-1})|$ is the speed of the object at the beginning of the jth time interval. Hence, for small enough Δt, $|v(t_{j-1})|\Delta t$ will give a good approximation of the distance the object will travel during the jth time interval. Thus if D represents the total distance the object travels from time $t = a$ to time $t = b$, then

$$D \approx \sum_{j=1}^{n} |v(t_{j-1})|\Delta t. \tag{4.8.1}$$

Moreover, we expect that as Δt decreases, or, equivalently, as n increases, this approximation should approach the exact value of D. That is, we should have

$$D = \lim_{n \to \infty} \sum_{j=1}^{n} |v(t_{j-1})|\Delta t. \tag{4.8.2}$$

Now the right-hand side of (4.8.1) is a Riemann sum (in particular, a left-hand rule sum) which approximates the definite integral

$$\int_a^b |v(t)|dt.$$

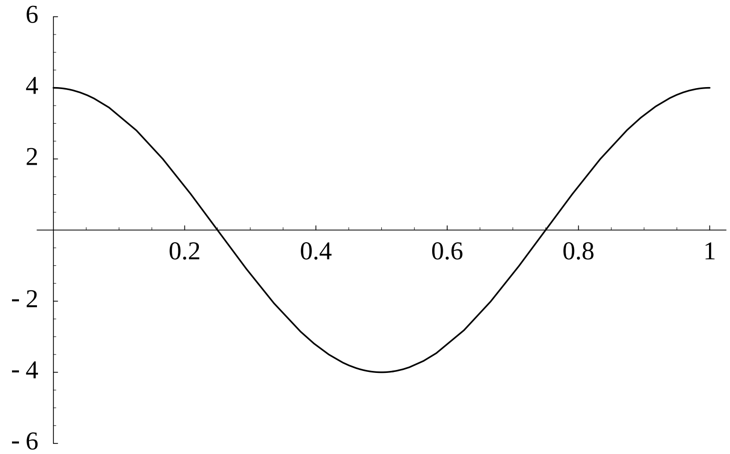

Figure 4.8.1 Graph of the velocity function $v(t) = 4\cos(2\pi t)$

Hence this integral is the value of the limit in (4.8.2), and so we have

$$D = \lim_{n\to\infty} \sum_{j=1}^{n} |v(t_{j-1})|\Delta t = \int_a^b |v(t)|dt. \tag{4.8.3}$$

Example Suppose an object is oscillating at the end of a spring so that its velocity at time t is given by $v(t) = 4\cos(2\pi t)$. Then the distance D traveled by the object from time $t = 0$ to time $t = 1$ is given by

$$D = \int_0^1 |4\cos(2\pi t)|dt = 4\int_0^1 |\cos(2\pi t)|dt.$$

Now

$$\cos(2\pi t) \geq 0 \text{ when } 0 \leq t \leq \frac{1}{4} \text{ or } \frac{3}{4} \leq t \leq 1,$$

and

$$\cos(2\pi t) \leq 0 \text{ when } \frac{1}{4} \leq t \leq \frac{3}{4}.$$

Hence

$$|\cos(2\pi t)| = \cos(2\pi t) \text{ when } 0 \leq t \leq \frac{1}{4} \text{ or } \frac{3}{4} \leq t \leq 1,$$

and

$$|\cos(2\pi t)| = -\cos(2\pi t) \text{ when } \frac{1}{4} \leq t \leq \frac{3}{4}.$$

Thus

$$\begin{aligned}\int_0^1 |\cos(2\pi t)|dt &= \int_0^{\frac{1}{4}} |\cos(2\pi t)|dt + \int_{\frac{1}{4}}^{\frac{3}{4}} |\cos(2\pi t)|dt + \int_{\frac{3}{4}}^{1} |\cos(2\pi t)|dt \\ &= \int_0^{\frac{1}{4}} \cos(2\pi t)dt - \int_{\frac{1}{4}}^{\frac{3}{4}} \cos(2\pi t)dt + \int_{\frac{3}{4}}^{1} \cos(2\pi t)dt \\ &= \frac{1}{2\pi}\sin(2\pi t)\Big|_0^{\frac{1}{4}} - \frac{1}{2\pi}\sin(2\pi t)\Big|_{\frac{1}{4}}^{\frac{3}{4}} + \frac{1}{2\pi}\sin(2\pi t)\Big|_{\frac{3}{4}}^{1} \\ &= \left(\frac{1}{2\pi} - 0\right) - \left(-\frac{1}{2\pi} - \frac{1}{2\pi}\right) + \left(0 + \frac{1}{2\pi}\right) \\ &= \frac{2}{\pi}.\end{aligned}$$

Hence

$$D = 4\int_0^1 |\cos(2\pi t)|dt = \frac{8}{\pi}.$$

Position

Again suppose v is continuous on $[a, b]$ and $v(t)$ represents, for $a \le t \le b$, the velocity at time t of an object moving on a line. Let $x(t)$ be the position of the object at time t and suppose we know the value of $x(a)$, the position of the object at the beginning of the time interval. It follows that

$$\dot{x}(t) = \frac{d}{dt}x(t) = v(t), \tag{4.8.4}$$

and so, by the Fundamental Theorem of Integral Calculus, for any t between a and b,

$$\int_a^t v(s)ds = x(t) - x(a). \tag{4.8.5}$$

Thus we have

$$x(t) = \int_a^t v(s)ds + x(a) \tag{4.8.6}$$

for $a \le t \le b$. In other words, if we are given the velocity of an object for every time t in the interval $[a, b]$ and the position of the object at time $t = a$, then we may use (4.8.6) to compute the position of the object at any time t in $[a, b]$.

Example As in the previous example, consider an object oscillating at the end of a spring so that its velocity is given by $v(t) = 4\cos(2\pi t)$. If $x(t)$ is the position of the object at time t and, initially, $x(0) = 3$, then

$$x(t) = \int_0^t 4\cos(2\pi s)ds + 3 = \frac{2}{\pi}\sin(2\pi s)\Big|_0^t + 3 = \frac{2}{\pi}\sin(2\pi t) + 3.$$

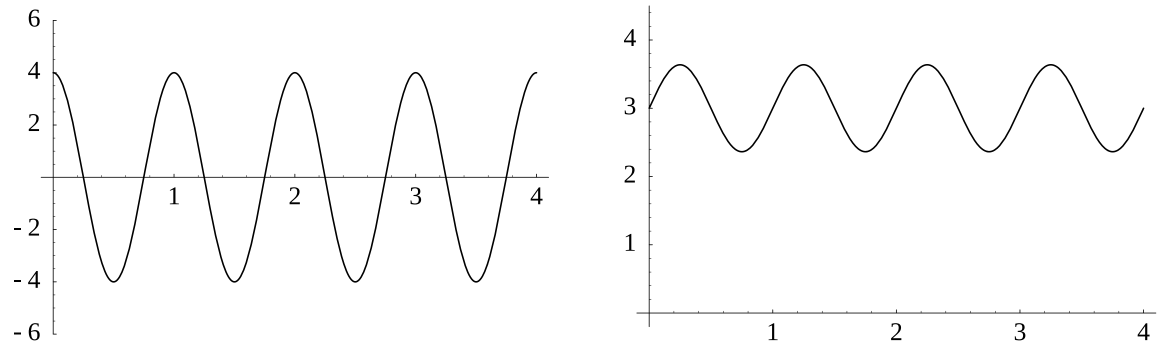

Figure 4.8.2 Velocity $v(t) = 4\cos(2\pi t)$ and position $x(t) = \dfrac{2}{\pi}\sin(2\pi t) + 3$

You should compare the graphs of the velocity function v and the position function x in Figure 4.8.2. Note that the object will oscillate between $3 - \frac{2}{\pi}$ and $3 + \frac{2}{\pi}$. In particular, the distance between these two extremes is $\frac{4}{\pi}$, and so the object will travel a distance of $\frac{8}{\pi}$ during a complete oscillation, in agreement with our computation in the previous example.

Example Suppose the velocity of an object at time t is given by $v(t) = 4\sin(t^2)$. If $x(t)$ is the position of the object at time t and its position at time 0 is $x(0) = -1$, then

$$x(t) = \int_0^t \sin(s^2)ds - 1.$$

However, unlike the previous example, there does not exist a simple antiderivative for v; hence, the best we can do is approximate $x(t)$ for a specified value of t using numerical integration. For example, we can compute numerically that

$$x(2) = \int_0^2 4\sin(s^2)ds - 1 = 2.219,$$

where we have rounded the result to the third decimal place. If we do this for enough points, we can plot the graph of x, as shown in Figure 4.8.3. Again, you should compare this graph with the graph of v, also shown in Figure 4.8.3.

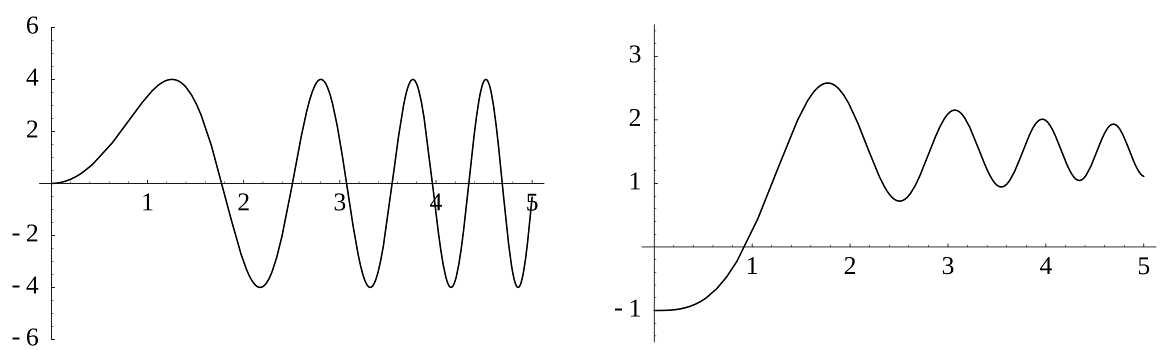

Figure 4.8.3 Velocity $v(t) = 4\sin(t^2)$ and position $x(t) = \int_0^t 4\sin(s^2)ds - 1$

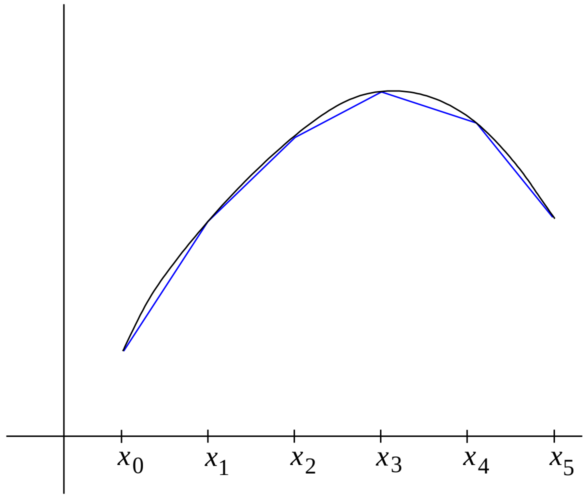

Figure 4.8.4 Approximating a curve with line segments

Length of a curve

Here we will consider the problem of finding the length of a curve which is the graph of some differentiable function. So suppose the function f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Let C be the graph of f and let L be the length of C. As we have done previously, we will first describe a method for finding good approximations to L. To begin, divide $[a, b]$ into n intervals of equal length

$$\Delta x = \frac{b-a}{n}$$

with endpoints $a = x_0 < x_1 < x_2 < \cdots < x_n = b$. For $j = 1, 2, 3, \ldots, n$, we can approximate the length of the piece of C lying over the jth interval by the distance between the endpoints of this piece, as shown in Figure 4.8.4. That is, since the endpoints of the jth piece are $(x_{j-1}, f(x_{j-1}))$ and $(x_j, f(x_j))$, we can approximate the length of the piece of C lying over the interval $[x_{j-1}, x_j]$ by

$$\sqrt{(x_j - x_{j-1})^2 + (f(x_j) - f(x_{j-1}))^2}.$$

Since $\Delta x = x_j - x_{j-1}$,

$$\begin{aligned} \sqrt{(x_j - x_{j-1})^2 + (f(x_j) - f(x_{j-1}))^2} &= \sqrt{(\Delta x)^2 + (f(x_j) - f(x_{j-1}))^2} \\ &= \sqrt{(\Delta x)^2 \left(1 + \frac{(f(x_j) - f(x_{j-1}))^2}{(\Delta x)^2}\right)} \qquad (4.8.7) \\ &= \Delta x \sqrt{1 + \left(\frac{f(x_j) - f(x_{j-1})}{\Delta x}\right)^2}. \end{aligned}$$

Hence, when n is large (equivalently, when Δx is small), a good approximation for L is given by

$$L \approx \sum_{j=1}^{n} \sqrt{1 + \left(\frac{f(x_j) - f(x_{j-1})}{\Delta x}\right)^2}\ \Delta x. \tag{4.8.8}$$

Moreover, we expect that

$$L = \lim_{n \to \infty} \sum_{j=1}^{n} \sqrt{1 + \left(\frac{f(x_j) - f(x_{j-1})}{\Delta x}\right)^2}\ \Delta x, \tag{4.8.9}$$

provided this limit exists. By the Mean Value Theorem, for each $j = 1, 2, 3, \ldots$, there exists a point c_j in the interval (x_{j-1}, x_j) such that

$$f'(c_j) = \frac{f(x_j) - f(x_{j-1})}{\Delta x}. \tag{4.8.10}$$

Hence

$$L = \lim_{n \to \infty} \sum_{j=1}^{n} \sqrt{1 + (f'(c_j))^2}\ \Delta x. \tag{4.8.11}$$

Now the sum in (4.8.11) is a Riemann sum for the integral

$$\int_a^b \sqrt{1 + (f'(x))^2}\ dx,$$

and so the limit, if it exists, converges to the value of this integral. Thus the length of C is given by

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\ dx. \tag{4.8.12}$$

Example Let L be the length of the graph of $f(x) = x^{\frac{3}{2}}$ on the interval $[0, 1]$, as shown in Figure 4.8.5. Then

$$f'(x) = \frac{3}{2}x^{\frac{1}{2}},$$

so

$$\begin{aligned} L &= \int_0^1 \sqrt{1 + \left(\frac{3}{2}x^{\frac{1}{2}}\right)^2}\ dx \\ &= \int_0^1 \sqrt{1 + \frac{9}{4}x}\ dx \\ &= \frac{8}{27}\left(1 + \frac{9}{4}x\right)^{\frac{3}{2}}\Bigg|_0^1 \\ &= \frac{13\sqrt{13} - 8}{27} = 1.4397, \end{aligned}$$

where we have rounded the result to four decimal places.

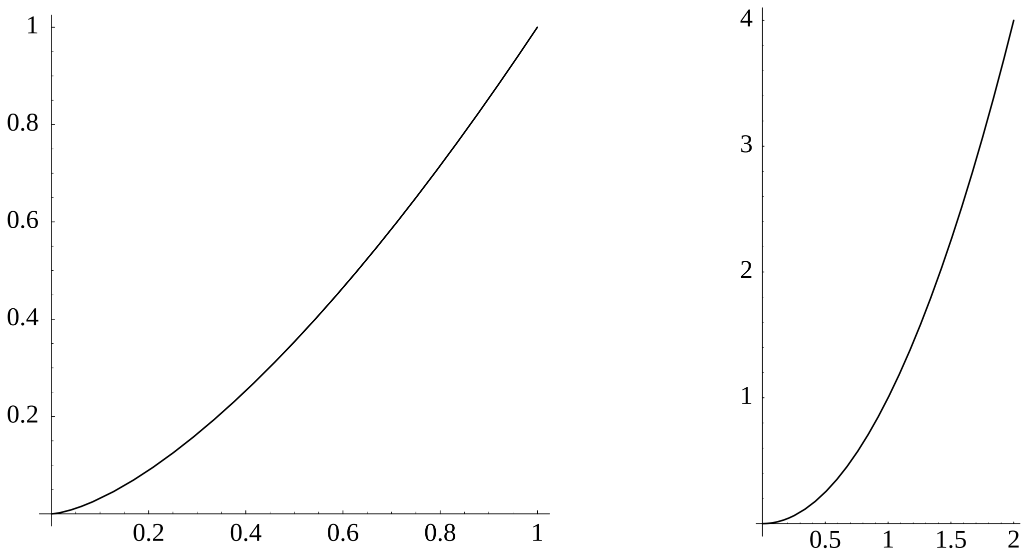

Figure 4.8.5 Graphs of $y = x^{\frac{3}{2}}$ and $y = x^2$

Example Let L be the length of the parabola $y = x^2$ from $(0,0)$ to $(2,4)$, as shown in Figure 4.8.5. Then

$$\frac{dy}{dx} = 2x,$$

so

$$L = \int_0^2 \sqrt{1 + (2x)^2}\, dx = \int_0^2 \sqrt{1 + 4x^2}\, dx.$$

At this point we do not have the techniques to evaluate this integral exactly using the Fundamental Theorem (although we will see such techniques in Chapter 6); however, we may use a computer algebra system to find that

$$L = \sqrt{17} + \frac{1}{4}\sinh^{-1}(4) = \sqrt{17} + \frac{1}{4}(\log(4 + \sqrt{17})),$$

where $\log(x)$ is the natural logarithm of x and $\sinh^{-1}(x)$ is the inverse hyperbolic sine of x. Since we will not study either of these functions until Chapter 6, we will use a numerical approximation to give us $L = 4.6468$ to four decimal places, the same answer we would obtain by using numerical integration to evaluate the integral.

Example To find the length L of one arch of the curve $y = \sin(x)$, as shown in Figure 4.8.6, we need to evaluate

$$L = \int_0^\pi \sqrt{1 + \cos^2(x)}\, dx,$$

an integral which is even more difficult than the one in the previous example. However, using numerical integration, we find that $L = 3.8202$ to four decimal places.

The last two examples illustrate some of the difficulties in finding the length of a curve. In general, the integrals involved in these problems require more sophisticated techniques than we have available at this time, and frequently require the use of numerical techniques.

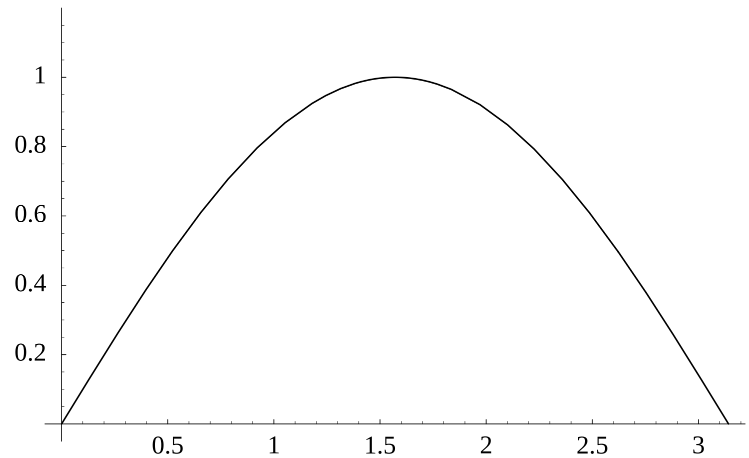

Figure 4.8.6 Graph of $y = \sin(x)$ over the interval $[0, \pi]$

Problems

1. For each of the following, assume that $v(t)$ is the velocity at time t of an object moving on a line and find the distance traveled by the object over the given time period.

   (a) $v(t) = 32t$ over $0 \leq t \leq 3$

   (b) $v(t) = -32t + 16$ over $0 \leq t \leq 3$

   (c) $v(t) = t^2 - t - 6$ over $0 \leq t \leq 2$

   (d) $v(t) = t^2 - t - 6$ over $0 \leq t \leq 4$

   (e) $v(t) = 2\sin(2t)$ over $0 \leq t \leq \pi$

   (f) $v(t) = 3\cos(2\pi t)$ over $0 \leq t \leq 2$

2. Suppose the velocity of a falling object is given by $v(t) = -32t$ feet per second. If the object is at a height of 100 feet at time $t = 0$, find the height of the object at an arbitrary time t.

3. Suppose $x(t)$ and $v(t)$ are the position and velocity, respectively, at time t of an object moving on a line. If $x(0) = 5$ and $v(t) = 3t^2 - 6$, find $x(t)$.

4. If an object of mass m is connected to a spring, pulled a distance x_0 away from its equilibrium position and released, then, ignoring the effects of friction, the velocity of the object at time t will be given by

$$v(t) = -x_0\sqrt{\frac{k}{m}}\sin\left(\sqrt{\frac{k}{m}}\,t\right),$$

   where k is a constant that depends on the strength of the spring. Find $x(t)$, the position of the object at time t.

5. Show that if $\dot{x}(t) = f(t)$ and f is continuous on $[a, b]$, then

$$x(t) = \int_a^t f(s)ds + x(a).$$

6. For each of the following, use the result from Problem 5 to find $x(t)$.

   (a) $\dot{x}(t) = 3t^3 + 6t - 17$ with $x(2) = 4$

   (b) $\dot{x}(t) = 3\cos(6t) - t$ with $x(0) = -1$

   (c) $\dot{x}(t) = \sin^2(2t)$ with $x(0) = 2$

   (d) $\dot{x}(t) = 3t^2 \sin(2t)$ with $x(0) = 0$

   (e) $\dot{x}(t) = \sqrt{1 + 2t}$ with $x(4) = 3$

7. Let $x(t)$, $v(t)$, and $a(t)$ be the height, velocity, and acceleration, respectively, at time t of an object of mass m in free fall near the surface of the earth. Let x_0 and v_0 be the height and velocity, respectively, of the object at time $t = 0$. If we ignore the effects of air resistance, the force acting on the body is $-mg$, where g is a constant ($g = 9.8$ meters per second, or 32 feet per second per second). Thus, by Newton's second law of motion,

$$-mg = ma(t),$$

   from which we obtain

$$a(t) = -g.$$

   Using Problem 5, show that

$$x(t) = -\frac{1}{2}gt^2 + v_0 t + x_0.$$

8. Suppose an object is projected vertically upward from a height of 100 feet with an initial velocity of 20 feet per second. Use Problem 7 to answer the following questions.

   (a) Find $x(t)$, the height of the object at time t.

   (b) At what time does the object reach its maximum height?

   (c) What is the maximum height reached by the object?

   (d) At what time will the object strike the ground?

9. For each of the following, find the length of the graph of the given function over the given interval.

   (a) $f(x) = 2x^{\frac{3}{2}}$ over $[0, 2]$

   (b) $f(x) = \sin(2x)$ over $\left[0, \frac{\pi}{2}\right]$

   (c) $g(x) = x^3$ over $[-1, 1]$

   (d) $g(t) = \tan(t)$ over $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

   (e) $f(t) = \sin^2(t)$ over $[0, \pi]$

   (f) $g(\theta) = \sin(\theta^2)$ over $[0, \sqrt{\pi}\,]$

10. A sheet of corrugated aluminum is to be made from a flat sheet of aluminum. Suppose a cross section of the corrugated sheet, when measured in inches, is in the shape of the curve

$$y = 2\sin\left(\frac{\pi}{4}t\right).$$

   Find the length of a flat sheet that would be needed to make a corrugated sheet that is 10 feet long.