Difference Equations to Differential Equations

Section 4.5

More Techniques of Integration

In the last section we saw how we could exploit our knowledge of the chain rule to develop a technique for simplifying integrals using suitably chosen substitutions. In this section we shall see how we can develop a second technique, called *integration by parts*, using the product rule. Outside of algebraic manipulation and the use of various functional identities, like the trigonometric identities, substitution and parts are the only basic techniques we have available to us for simplifying the process of evaluating an integral.

Example Suppose we wish to find $\int x\cos(x)dx$. Since

$$\int \cos(x)dx = \sin(x) + c,$$

we might make an initial guess of $F(x) = x\sin(x)$ for an antiderivative of $f(x) = x\cos(x)$. But, of course, differentiation of F, using the product rule, yields

$$F'(x) = x\cos(x) + \sin(x),$$

which differs from the desired result, $f(x)$, by the term $\sin(x)$. However, since

$$\int \sin(x)dx = -\cos(x) + c,$$

we can obtain an antiderivative of $f(x)$ by adding on the term $\cos(x)$ to $F(x)$. That is,

$$G(x) = x\sin(x) + \cos(x)$$

is an antiderivative of $f(x)$ since the derivative of $\cos(x)$ will cancel the $\sin(x)$ term in $F'(x)$. Explicitly,

$$G'(x) = x\cos(x) + \sin(x) - \sin(x) = x\cos(x).$$

Thus

$$\int x\cos(x)dx = x\sin(x) + \cos(x) + c.$$

In general, suppose f and g are differentiable functions and we want to evaluate

$$\int f(x)g'(x)dx. \tag{4.5.1}$$

For example, in our previous example we would have $f(x) = x$ and $g(x) = \sin(x)$. From the product rule we know that

$$\frac{d}{dx}f(x)g(x) = f(x)g'(x) + g(x)f'(x). \tag{4.5.2}$$

Thus, integrating both sides of (4.5.2), we have

$$\int \frac{d}{dx}f(x)g(x) = \int f(x)g'(x)dx + \int g(x)f'(x)dx, \tag{4.5.3}$$

from which it follows that

$$f(x)g(x) = \int f(x)g'(x)dx + \int g(x)f'(x)dx. \tag{4.5.4}$$

Rearranging (4.5.4) gives us

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx. \tag{4.5.5}$$

Applying (4.5.5) to our example, with $f(x) = x$ and $g(x) = \sin(x)$, we have

$$\int x\cos(x)dx = x\sin(x) - \int \sin(x)dx = x\sin(x) + \cos(x) + c.$$

In effect, using (4.5.5), we have replaced the problem of evaluating $\int x\cos(x)dx$ with the simpler problem of evaluating $\int \sin(x)dx$. In general, the success of this method always depends on the integral

$$\int g(x)f'(x)dx \tag{4.5.6}$$

being easier to evaluate than the integral

$$\int f(x)g'(x)dx. \tag{4.5.7}$$

It is common with this technique to let $u = f(x)$ and $v = g(x)$ along with the notation, as we did with substitution,

$$dv = g'(x)dx \tag{4.5.8}$$

and

$$du = f'(x)dx. \tag{4.5.9}$$

With this notation, (4.5.5) becomes

$$\int u dv = uv - \int v du, \tag{4.5.10}$$

the standard form for what is known as integration by parts.

Example To evaluate the integral $\int x \sin(x)$ by parts, we must first make a choice for u and dv. Here we might choose

$$u = x \quad dv = \sin(x)dx.$$

It follows then that $du = dx$. However, there are many possible choices for v; all that we require is that the derivative of v must be $\sin(x)$. The simplest choice is to take $v = -\cos(x)$. Then we have, applying (4.5.10),

$$\int x \sin(x)dx = -x\cos(x) + \int \cos(x)dx = -x\cos(x) + \sin(x) + c.$$

Example To evaluate the integral $\int x^2 \cos(2x)dx$, we might choose

$$u = x^2 \quad dv = \cos(2x)dx,$$

from which we obtain

$$du = 2xdx \quad v = \frac{1}{2}\sin(2x).$$

Thus

$$\int x^2 \cos(2x)dx = \frac{1}{2}x^2 \sin(2x) - \int x \sin(2x)dx.$$

This time we do not immediately know the value of the integral on the right, but we know we can find it using integration by parts. Namely, to evaluate $\int x \sin(2x)dx$, we let

$$\begin{array}{ll} u = x & dv = \sin(2x)dx \\ du = dx & v = -\frac{1}{2}\cos(2x). \end{array}$$

Then

$$\int x \sin(2x)dx = -\frac{1}{2}x\cos(2x) + \frac{1}{2}\int \cos(2x)dx = -\frac{1}{2}x\cos(2x) + \frac{1}{4}\sin(2x) + c.$$

Hence

$$\int x^2 \cos(2x)dx = \frac{1}{2}x^2 \sin(2x) + \frac{1}{2}x\cos(2x) - \frac{1}{4}\sin(2x) + c.$$

The key to success with integration by parts is in the choice of the parts, u and dv. For example, we saw in an example that the choices

$$\begin{array}{ll} u = x & dv = \sin(x)dx \\ du = dx & v = -\cos(x) \end{array}$$

work well for evaluating $\int x\sin(x)dx$. Alternatively, we could have chosen

$$\begin{aligned} u &= \sin(x) & dv &= xdx \\ du &= \cos(x)dx & v &= \frac{1}{2}x^2, \end{aligned}$$

which would yield

$$\int x\sin(x)dx = \frac{1}{2}x^2\sin(x) - \frac{1}{2}\int x^2\cos(x)dx.$$

All of this is correct, but useless (at least for our present purpose) since the resulting integral on the right is more complicated than the integral with which we started. If we had started to work the problem this way, we would probably stop at this point and rethink our strategy.

Example In using integration by parts to evaluate a definite integral, we must remember to evaluate all the pieces of the resulting antiderivative. For example, to evaluate

$$\int_0^{\frac{\pi}{3}} 4x\cos(3x)dx,$$

we might choose

$$\begin{aligned} u &= 4x & dv &= \cos(3x)dx \\ du &= 4dx & v &= \frac{1}{3}\sin(3x). \end{aligned}$$

Then

$$\begin{aligned} \int_0^{\frac{\pi}{3}} 4x\cos(3x)dx &= \frac{4}{3}x\sin(3x)\Big|_0^{\frac{\pi}{3}} - \frac{4}{3}\int_0^{\frac{\pi}{3}} \sin(3x)dx \\ &= (0-0) + \frac{4}{9}\cos(3x)\Big|_0^{\frac{\pi}{3}} \\ &= -\frac{4}{9} - \frac{4}{9} \\ &= -\frac{8}{9}. \end{aligned}$$

Example Although integration by parts is most frequently of use when integrating functions involving transcendental functions, such as the trigonometric functions, there are other times when the technique may be used. For example, to compute

$$\int_0^1 x(1+x)^{10}dx,$$

we could use

$$\begin{aligned} u &= x & dv &= (1+x)^{10}dx \\ du &= dx & v &= \frac{1}{11}(1+x)^{11}. \end{aligned}$$

Then

$$\begin{aligned}\int_0^1 x(1+x)^{10}dx &= \frac{1}{11}x(1+x)^{11}\Big|_0^1 - \frac{1}{11}\int_0^1 (1+x)^{11}dx\\ &= \frac{2048}{11} - \frac{1}{132}(1+x)^{12}\Big|_0^1\\ &= \frac{2048}{11} - \left(\frac{4096}{132} - \frac{1}{132}\right)\\ &= \frac{6827}{44}.\end{aligned}$$

Notice that we could also evaluate this integral using the substitution $u = 1 + x$.

Miscellaneous examples

The techniques of substitution and parts are often useful for putting an integral into a form that can be readily evaluated by the Fundamental Theorem of Integral Calculus. The next several examples illustrate how basic trigonometric identities are also useful for rewriting integrals in more easily evaluated forms.

Example To evaluate $\int \sin^2(x)dx$, we may use the identity

$$\sin^2(x) = \frac{1-\cos(2x)}{2} \tag{4.5.11}$$

(see Problem 5, Section 2.2). Then

$$\int \sin^2(x)dx = \frac{1}{2}\int (1-\cos(2x))dx = \frac{1}{2}x - \frac{1}{4}\sin(2x) + c.$$

Example Similarly, to evaluate

$$\int_0^{\frac{\pi}{4}} \cos^2(2t)dt,$$

we use the identity

$$\cos^2(x) = \frac{1+\cos(2x)}{2}. \tag{4.5.12}$$

Then

$$\begin{aligned}\int_0^{\frac{\pi}{4}} \cos^2(2t)dt &= \frac{1}{2}\int_0^{\frac{\pi}{4}} (1+\cos(4t))dt\\ &= \frac{1}{2}t\Big|_0^{\frac{\pi}{4}} + \frac{1}{8}\sin(4t)\Big|_0^{\frac{\pi}{4}}\\ &= \frac{\pi}{8}.\end{aligned}$$

Example To evaluate $\int \sin^2(x)\cos^2(x)dx$, the identity

$$\sin(x)\cos(x) = \frac{1}{2}\sin(2x) \tag{4.5.13}$$

is useful (see Problem 4, Section 2.2.). From it we obtain

$$\begin{aligned}\int \sin^2(x)\cos^2(x)dx &= \int (\sin(x)\cos(x))^2 dx \\ &= \int \left(\frac{1}{2}\sin(2x)\right)^2 dx \\ &= \frac{1}{4}\int \sin^2(2x)dx \\ &= \frac{1}{8}\int (1-\cos(4x))dx \\ &= \frac{1}{8}x - \frac{1}{32}\sin(4x) + c.\end{aligned}$$

Example To evaluate $\int \sin^3(x)dx$ we may use the identity

$$\sin^2(x) = 1 - \cos^2(x)$$

to write

$$\sin^3(x) = \sin^2(x)\sin(x) = (1-\cos^2(x))\sin(x).$$

Then the substitution

$$\begin{aligned}u &= \cos(x) \\ du &= -\sin(x)dx\end{aligned}$$

gives us

$$\begin{aligned}\int \sin^3(x)dx &= \int (1-\cos^2(x))\sin(x)dx \\ &= -\int (1-u^2)du \\ &= -u + \frac{1}{3}u^3 + c \\ &= -\cos(x) + \frac{1}{3}\cos^3(x) + c.\end{aligned}$$

This manipulation is useful in evaluating any integral of the form $\int \sin^n(x)dx$ or, in a similar fashion, $\int \cos^n(x)dx$, provided n is a positive odd integer.

Example As a final example, note that the identity

$$\tan^2(x) = \sec^2(x) - 1$$

(see Problem 3, Section 2.2) is useful in evaluating

$$\int_0^{\frac{\pi}{4}} \tan^2(x)dx.$$

Namely,

$$\begin{aligned}\int_0^{\frac{\pi}{4}} \tan^2(x)dx &= \int_0^{\frac{\pi}{4}} (\sec^2(x) - 1)dx \\ &= \tan(x)\Big|_0^{\frac{\pi}{4}} - x\Big|_0^{\frac{\pi}{4}} \\ &= 1 - \frac{\pi}{4}.\end{aligned}$$

This concludes our discussion of techniques of integration. As we noted above, there are basically only two techniques for evaluating indefinite integrals, substitution and parts, and even these rely on an ability to reduce a given integral to a form where an antiderivative is recognizable. Hence the situation is not nearly as straightforward as it was for finding derivatives and best affine approximations. For this reason, in the past tables of indefinite integrals were compiled to aid in the evaluation of integrals; when faced with an integral more involved than the basic ones we have investigated in these last two sections, one could hope to find it, or one related to it through a substitution or an integration by parts, in a table. For the most part, tables of integrals have been replaced by computer programs, such as computer algebra systems, which are capable of finding antiderivatives symbolically. Such programs are then able to evaluate definite integrals exactly using the Fundamental Theorem. Although these programs are immensely useful and are an everyday tool for those working with applications of mathematics, one must use them with care. In particular, whenever possible, you should check your answer for reasonableness. Moreover, there are integrals which the system will not be able to evaluate symbolically, either because the given integral is beyond the capabilities of the system, or because a symbolic answer does not even exist. In such cases, one must, of necessity, fall back on numerical approximation techniques.

Problems

1. Evaluate the following indefinite integrals.

   (a) $\displaystyle\int 3x\sin(x)dx$

   (b) $\displaystyle\int 2x\cos(5x)dx$

   (c) $\displaystyle\int 4x\sin(3x)dx$

   (d) $\displaystyle\int x^2\cos(3x)dx$

   (e) $\displaystyle\int 2x^2\sin(4x)dx$

   (f) $\displaystyle\int x^3\cos(x)dx$

   (g) $\displaystyle\int 3x^3\sin(2x)dx$

   (h) $\displaystyle\int x\sqrt{1+x}\ dx$

2. Evaluate the following definite integrals.

   (a) $\displaystyle\int_0^{\pi} 4x\sin(x)dx$ (b) $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 3x\cos(2x)dx$

   (c) $\displaystyle\int_0^{\frac{\pi}{3}} 2t\sin(3t)dt$ (d) $\displaystyle\int_0^{\frac{\pi}{2}} x^2\cos(x)dx$

   (e) $\displaystyle\int_0^{\frac{\pi}{4}} 2x^2\sin(2x)dx$ (f) $\displaystyle\int_0^{\frac{\pi}{4}} z^3\cos(4z)dz$

3. Evaluate the following indefinite integrals.

   (a) $\displaystyle\int \sin^2(2x)dx$ (b) $\displaystyle\int \cos^2(3t)dt$

   (c) $\displaystyle\int 5\sin^2(2t)\cos^2(2t)dt$ (d) $\displaystyle\int \sin^3(3x)dx$

   (e) $\displaystyle\int 6\cos^3(2z)dz$ (f) $\displaystyle\int \sin^5(t)dt$

   (g) $\displaystyle\int \cos^5(2x)dx$ (h) $\displaystyle\int \tan^2(3\theta)d\theta$

4. Evaluate the following definite integrals.

   (a) $\displaystyle\int_0^{\pi} \sin^2(x)dx$ (b) $\displaystyle\int_0^{\frac{\pi}{4}} \cos^2(2t)dt$

   (c) $\displaystyle\int_0^{\frac{\pi}{2}} 3\sin^2(z)\cos^2(z)dz$ (d) $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^3(t)dt$

   (e) $\displaystyle\int_0^{\pi} \sin^3(3t)dt$ (f) $\displaystyle\int_{-\frac{\pi}{8}}^{\frac{\pi}{8}} \tan^2(2t)dt$

5. Evaluate the following integrals using a computer algebra system.

   (a) $\displaystyle\int \cos^6(x)dx$ (b) $\displaystyle\int \sin^2(2t)\cos^4(2t)dt$

   (c) $\displaystyle\int \sin^4(2t)\cos^4(3t)dt$ (d) $\displaystyle\int \sec^4(3x)dx$

   (e) $\displaystyle\int_{-1}^{1} \sqrt{1-x^2}\,dx$ (f) $\displaystyle\int_0^{1} x^2\sqrt{1-x^2}\,dx$

   (g) $\displaystyle\int_0^{2\pi} \sin^8(2t)dt$ (h) $\displaystyle\int_0^{\frac{\pi}{4}} \tan^6(t)dt$

6. Evaluate the following integrals with any method at your disposal.

   (a) $\displaystyle\int_{-\pi}^{\pi} \sin^4(x)dx$ (b) $\displaystyle\int_1^{\pi} \frac{\sin(x)}{x}\,dx$

   (c) $\displaystyle\int_0^{5} \sin(3x^2)dx$ (d) $\displaystyle\int_0^{1} \sqrt{1+x^2}\,dx$

(e) $\displaystyle\int_1^2 \frac{1}{x}\,dx$

(f) $\displaystyle\int_{-2}^{-1} \frac{1}{t}\,dt$

(g) $\displaystyle\int_0^{\pi} \sqrt{5-3\sin^2(t)}\,dt$

(h) $\displaystyle\int_0^{2\pi} \frac{1}{\sqrt{1+\sin^2(t)}}\,dt$

7. If a pendulum of length b is held, at rest, at an angle α from the perpendicular, $0 < \alpha < \pi$, and then released, its period T, the time required for one complete oscillation, is given by

$$T = 4\sqrt{\frac{b}{g}}\int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-k^2\sin^2(\varphi)}}\,d\varphi,$$

where $g = 980$ cm/sec^2 (the acceleration due to gravity) and $k = \sin\left(\frac{\alpha}{2}\right)$.

(a) Find the period of a pendulum of length 50 centimeters which is released initially from an angle of $\alpha = \frac{\pi}{3}$.

(b) Repeat (a) for $\alpha = \frac{\pi}{4}, \frac{\pi}{6}, \frac{\pi}{50}$, and $\frac{\pi}{100}$.

(c) In Section 2.2 we noted that for small values of α, if $x(t)$ represents the angle the pendulum makes with the perpendicular at time t, then, to a good approximation,

$$x(t) = \alpha\cos\left(\sqrt{\frac{g}{b}}t\right).$$

Thus, in this approximation, x has a period of $2\pi\sqrt{\frac{b}{g}}$. For a pendulum of length 50 centimeters, compare this result with your results in parts (a) and (b).

(d) For a pendulum of length 50 centimeters, graph T as a function of α for $-\frac{\pi}{4} \leq \alpha \leq \frac{\pi}{4}$.