Sine and Cosine

Radians

For the first eight lectures of this class, we restricted our attention to polynomial functions of one form or another. We now move on to non-polynomial functions. The first non-polynomial functions which we will study are the sine function and the cosine function. Before we can study the sine and cosine functions, we need to talk a little about radians.

Draw an xy-plane, and then draw a circle of radius r centered at the origin. Mark the origin with the letter O and the place where the circle intersects the positive x-axis with the letter A. Now draw a straight line emanating from the origin out into the first quadrant. Mark the point where this line intersects the circle with the letter B (both of the coordinates of the point B should be positive). We now have an arc of the circle AB, which is connected to the origin by the line segments OA and OB, which both have length r. At the origin, we have the angle $\angle AOB$.

Let us denote the length of the arc of AB by s. The ratio of s to r, of the length of the arc of AB to the radius of the circle, is called the number of radians of the angle $\angle AOB$. We will denote the number of radians by the lower case Greek letter theta:

$$\theta = \frac{s}{r}.$$

The number of radians of an angle does not depend on the radius of the circle. To see this, extend the line segments OA and OB out from A and B respectively so that the resulting line segments have length $2r$. Call the new endpoints C and D respectively. The line segments OC and OD are radii of the circle centered at the origin of radius $2r$, so C and D are connected by an arc of this circle, the arc CD. If you were to take a piece of string and measure the length of the arc CD, you would find that it is twice that of the length of the arc AB, or, to write this algebraically, the length is $2s$. Now, we should all agree that the angle $\angle AOB$ and the angle $\angle COD$ are the same angle. Let us see if they have the same number of radians:

$$\theta = \frac{2s}{2r} = \frac{s}{r}.$$

They do have the same number of radians, and, if we were to repeat this with any angle and any two different lengths of radii, we would get the same result: the number of radians of an angle is an invariant, a fixed quantity, of that angle. It is a measure of that angle.

There is another common measure for angles, and that measure is degrees. How are degrees and radians related to each other? To see the relationship, consider the $360°$ angle, which is the angle we get by doing exactly one revolution counterclockwise around the circle. Given a circle of radius r, the length of the arc defined by a $360°$ angle is precisely the circumference of the circle, which is $2\pi r$. So the number of radians in any $360°$ angle is

$$\theta = \frac{2\pi r}{r} = 2\pi.$$

The ratio of the number of radians in a $360°$ angle to the number of degrees is

$$\frac{2\pi}{360} = \frac{\pi}{180}.$$

This ratio is a constant for all angles: if you want to convert the measure of an angle from degrees to radians, you multiply the number of degrees by $\frac{\pi}{180}$, and, if you want to convert from radians to degrees, you multiply the number of radians by $\frac{180}{\pi}$. A list of common angle measures in both degrees and radians is given below:

Degrees	**Radians**
0	0
30	$\frac{\pi}{6}$
45	$\frac{\pi}{4}$
60	$\frac{\pi}{3}$
90	$\frac{\pi}{2}$

Degrees	**Radians**
120	$\frac{2\pi}{3}$
135	$\frac{3\pi}{4}$
150	$\frac{5\pi}{6}$
180	π
270	$\frac{3\pi}{2}$
360	2π

It is absolutely necessary that you memorize the number of radians in each of the angles in the tables above.

The final point to make about radians is the concept of angles with measures greater than 2π and angles with negative measure. First, we remember that the number of radians in an angle is equal to the length of the arc of the circle determined by that angle divided by the radius of the circle. To find an angle of θ radians, where θ is positive, we begin at where the circle intersects the positive x-axis, and we wind counterclockwise around the circle until the arc we have followed has length $|r\theta|$. The key point is that we are allowed to wind around the circle as many times as necessary in order to get an arc of the proper length. So, for example, for an angle of 6π radians, we would wind around the circle exactly three times, because to go around the circle once is 2π radians, and the angle in question is three times as large. Likewise, for an angle of 3π radians, we would wind around the circle once, and then wind half way around again, stopping where we would stop for π radians (or 180 degrees). The angles π and 3π are distinct, they are not the same, but they do end at the same point on the circle: one just took longer to get there than the other.

As for negative measure for an angle: we can find an angle of θ radians, where θ is negative, by winding around the circle clockwise, instead of counterclockwise. So, to find an angle of $-\frac{\pi}{2}$ radians, we wind one quarter the way around the circle in the clockwise direction, starting at the positive x-axis. To find an angle of -6π radians, we again start at the positive x-axis and wind around the circle exactly three times clockwise. We stop at the same point as we would for 0 radians, or 2π radians, or $k\pi$ radians, where k is any multiple of 2 (positive, negative, or zero), but, to reiterate, these are all distinct angles.

Sine and Cosine

We now define the sine and cosine of an angle. We begin with a drawing a right triangle ABC, with $\angle C$ being the right angle. Consider the acute angle $\angle A$. The opposite of $\angle A$ is the line segment BC, the adjacent is the line segment AC, and the hypothenuse of the triangle, opposite the right angle, is the line segment AB. The sine of the angle $\angle A$ is defined to be the length of its opposite divided by the length of the hypothenuse:

$$\text{sine of } \angle A = \frac{\text{length of } BC}{\text{length of } AB}.$$

The cosine of the angle $\angle A$ is defined to be the length of its adjacent divided by the length of the hypothenuse:

$$\text{cosine of } \angle A = \frac{\text{length of } AC}{\text{length of } AB}.$$

The sine and cosine of an angle is an invariant of the measure of the angle, that is, all angles of the same number of radians have the same sine and the same cosine. We denote the sine of an angle of θ radians by $\sin\theta$, and the cosine of an angle of θ radians by $\cos\theta$.

There is one final point to remember about right triangles, and that is the Pythagorean Theorem, which states that the square of the length of the hypothenuse is equal to the sum of the squares of the lengths of the other two sides:

$$(\text{length of } AB)^2 = (\text{length of } AC)^2 + (\text{length of } BC)^2.$$

These concepts of sine and cosine of an angle are currently only defined for acute angles, that is, positive angles of less than $\frac{\pi}{2}$ radians. We now extend the notion of sine and cosine to all angles. We begin, as usual, by drawing a circle centered at the origin. This time, we make the radius of the circle equal to 1. Pick a point on the circle inside the first quadrant, and draw the radius of the circle which connects the origin, point O, to the point on the circle, point A. Now draw the vertical line segment which connects the point A to the positive x-axis, and mark the point at which it touches the positive x-axis as point B. The angle $\angle OBA$ is a right angle. We are interested in finding the sine and cosine of the angle $\angle AOB$, the angle at the origin. Mark this angle as having a measure of θ radians. Then, according to our formulas for sine and cosine, and the fact that the hypothenuse of the right triangle AOB is a radius of the circle and thus has length 1, we find that

$$\sin\theta = \text{length of } AB \qquad \text{and} \qquad \cos\theta = \text{length of } OA.$$

Now here is the key point: look at the coordinates of the point A. The x-coordinate of A is precisely the length of OA, and the y-coordinate of A is precisely the length of AB. So we can rewrite the formulae above as

$$\sin\theta = y\text{-coordinate of } A \qquad \text{and} \qquad \cos\theta = x\text{-coordinate of } A.$$

We now have a way to extend the idea of sine and cosine to all angles, in the following way: to find the sine and cosine of θ, first, draw a unit circle centered at the origin. Now, starting at $(1, 0)$, the point where the unit circle intersects the positive x-axis, wind around the circle (counterclockwise for a positive angle, clockwise for a negative angle) until the arc we have followed has length $|\theta|$. Find the coordinates of the endpoint of this arc. We define the $\sin\theta$ to be the y-coordinate of the endpoint, and $\cos\theta$ to be the x-coordinate of the endpoint.

You must memorize the sine and cosine of several important angles, in particular, all angles of the form $\frac{n\pi}{6}$, $\frac{n\pi}{3}$, and $\frac{n\pi}{4}$, where n is any integer. All angles of this form between 0 and 2π, along with their cosines and sines (notice the order), are found below:

θ	$\cos\theta$	$\sin\theta$
0	1	0
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{2}$	0	1
$\frac{2\pi}{3}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{3\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\frac{5\pi}{6}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
π	-1	0

θ	$\cos\theta$	$\sin\theta$
$\frac{7\pi}{6}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$
$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$
$\frac{4\pi}{3}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$
$\frac{3\pi}{2}$	0	-1
$\frac{5\pi}{3}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$
$\frac{7\pi}{4}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$
$\frac{11\pi}{6}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$
2π	1	0

To get any other angle of this form, just repeat the pattern above. The easiest way to remember this pattern of sines and cosines is to draw out a unit circles with all of these angles shown, and we encourage you to do this.

The last point to make today is the relationship between $\sin\theta$ and $\cos\theta$. In the case of acute angles, we see that $\sin\theta$ and $\cos\theta$ are the lengths of the legs of a right triangle with a hypothenuse of length 1. Therefore, applying the Pythagorean Theorem, we see that (notice where the exponents are):

$$\sin^2\theta + \cos^2\theta = 1.$$

This relationship, that the sum of the squares of sine and cosine is equal to 1, applies to all angles, not just acute angles. It is the most important equation in trigonometry, and you should make sure that you always remember it. Test it on the angles in the table above to see that it works.