Positive Power Functions

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In today's lecture, we continue our theme of studying polynomials and the terms which comprise polynomials. A positive power function is a function of the form $f(x) = x^n$, where n is a natural number. You should become familiar with seeing notation like this, where the exponent is not given explicitly, but instead is represented by a letter. Here, we mean that we are studying the function $f(x) = x, x^2, x^3, x^4, ...$ as a family.

Suppose we were to graph this family of functions all on the same graph. Specifically, let us graph x, x^2, x^3, and x^4 using the following numerical tables as a guide (we leave out the table for the function x):

x	x^2
-1.5	2.25
-1	1
-0.5	0.25
0	0
0.5	0.25
1	1
1.5	2.25

x	x^3
-1.5	-3.375
-1	-1
-0.5	-0.125
0	0
0.5	0.125
1	1
1.5	3.375

x	x^4
-1.5	5.0625
-1	1
-0.5	0.0625
0	0
0.5	0.0625
1	1
1.5	5.0625

When we sketch out these graphs, we see the following traits of the graphs of positive power functions:

- The graphs of positive power functions come in two basic shapes. If n is even, then the graph of x^n has an axis of symmetry, and that is the vertical line $x = 0$, the y-axis. If n is odd, then x^n has a point of symmetry, which is the origin. This means that if the point (x, y) is in the graph of x^n, then the point $(-x, -y)$ is also in the graph. We will discuss these two types of symmetries more later in the lecture.
- For all values of n, x^n equals 0 when $x = 0$, and x^n equals 1 when $x = 1$. So the graphs of all positive power functions intersect at the origin and at the point $(1, 1)$.
- When x is between 0 and 1, the larger n is, the smaller x^n is. You can observe this phenomenon but studying the values of x, x^2, x^3, x^4 for $x = 0.5$ in the tables above. This has the effect that, if m and n are two natural numbers and $m < n$, then between $x = 0$ and $x = 1$, the graph of x^m is above the graph of x^n. We also see that, as n gets larger, the graph of x^n looks more and more like it is following the x-axis and the vertical line $x = 1$, so that it almost looks like it has a corner in it.
- When x is greater than 1, the larger n is, the larger x^n is. Think of the powers of any natural number greater than 1 to see this. This means that, if m and n are natural numbers and $m < n$, then when $x > 1$, the graph of x^m is below that of x^n. We also observe that, the larger n is, the more the graph of x^n for $x > 1$ looks like the vertical line $x = 1$.

Derivatives of Positive Power Functions

Now let $f(x) = x^n$, where n is some natural number. The derivative of $f(x)$ at $x = p$ is

$$\frac{\mathrm{d}f}{\mathrm{d}x}(p) = np^{n-1}.$$

In particular:

- If $f(x) = x$, then $\frac{\mathrm{d}f}{\mathrm{d}x}(p) = 1$. This should make sense: in this case, $f(x)$ is a linear function. Recall that the derivative of a function at a point is the slope of the tangent line to the graph of that function at that point. The graph of a linear function is a line, and, when you think about it, the tangent line to a line is itself. So the derivative of a linear function is its own slope. In this case, that slope is 1.
- If $f(x) = x^2$, then the formula above tells us that $\frac{\mathrm{d}f}{\mathrm{d}x}(p) = 2p$. This is consistent with what we already know, because here $f(x)$ is a quadratic function, with $a = 1$ and $b = 0$, and applying the formula for the derivative of a quadratic function, we get the same result as above.

- If $f(x) = x^3$, then $\frac{\mathrm{d}f}{\mathrm{d}x}(p) = 3p^2$. Now $f(x)$ is a cubic function, with $a = 1$, $b = 0$, and $c = 0$, and here too we would have gotten the same result had we used the formula for the derivative of a cubic.

As an example of using this formula, let us find the derivatives of all of the positive power function at $x = 1$. Since we are working with all of the positive power function, let us use the notation $f(x) = x^n$ instead of specific values of n. Then applying the formula for the derivative, we get that

$$\frac{\mathrm{d}f}{\mathrm{d}x}(1) = n \cdot 1^{n-1} = n \cdot 1 = n.$$

So the slope of the tangent line to the graph of x^n at $x = 1$ is n. This means that, as n gets larger, so does the slope of that tangent line. Does this agree with our observations of how the graphs of positive power functions look when compared to each other?

Even and Odd Functions

We now return to the issue of the symmetry we found in the graphs of positive power functions. Remember, when n is even, the graph of $f(x) = x^n$ has an axis of symmetry at the y-axis. This axis of symmetry exists specifically because $(-x)^n = x^n$ when n is even. Another way to write this is that is by using function notation: $f(-x) = f(x)$. It turns out that many functions have the property that $f(-x) = f(x)$. Since the most important of these are the positive power functions when n is even, the name we give to functions with this property is even functions. The main characteristic of graphs of even functions is that they all have an axis of symmetry at the y-axis. It is also true that if the graph of a function has an axis of symmetry at the y-axis, then that function is an even function.

We also observed that, when n is odd, the graph of $f(x) = x^n$ has a point of symmetry at the origin. This point of symmetry exists because $(-x)^n = -x^n$ when n is odd (if you do not see this, try computing both sides of this equation using specific values of x and n). We can also write this relationship using function notation: $f(-x) = -f(x)$. Just as in the previous case, there are many functions with the property $f(-x) = -f(x)$, and for because the most prominent examples of functions like these are the positive power functions when n is odd, the name for functions with this property is odd functions. The graphs of odd functions have the characteristic that they have point symmetry about the origin and, specifically, they pass through the origin. In other words, if f is an odd function, then $f(0) = 0$. Can you see why this must be true? We also have that, if the graph of a function has point symmetry about the origin, then that function is an odd function.

The Derivative as a Function and Higher Derivatives

We now return to studying the derivative. In all of the functions we have described so far, we have talked about the notion of the derivative at a point. You may have noticed, however, that we do not need to think about the derivative as being at one point and one point only. We can think of the derivative of $f(x)$ as being another function: we assign to every x the derivative of $f(x)$ at that x value. We write this new function as $\frac{\mathrm{d}f}{\mathrm{d}x}(x)$, so simply $\frac{\mathrm{d}f}{\mathrm{d}x}$, and instead of calling this the derivative function, we simply call it the derivative. In the future, if we do not specify a point at which we are taking the derivative, then we are taking about the derivative as a function. If we do specify a point, then we are taking about the derivative at that point, which is simply a number.

As a example of the derivative as a function, take $f(x) = 2x^2 + 3x - 5$. Then

$$\frac{\mathrm{d}f}{\mathrm{d}x}(x) = 4x + 3.$$

Notice that all we did was apply the formula for the derivative of a quadratic, and instead of substituting a number for p, we simply replaced p with x. Also notice that the derivative of $f(x)$, which is a quadratic function, is a linear function. This is a general phenomenon: the derivatives of all quadratic functions are linear functions. Likewise, the derivative of a cubic function is a quadratic function, and the derivative of a linear function is a constant function. Can you see a pattern developing?

The concept of the derivative as a function leads us to another idea: taking the derivative of a derivative. The derivative of the derivative of the function $f(x)$ is called the second derivative of $f(x)$. In this context, the derivative of $f(x)$ is called the first derivative. We write the second derivative as $\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}$. Pay special attention to where the number 2 appears in the expression for the second derivative: the placement in the denominator is different that that in the numerator.

Let us find the second derivative of the function $f(x) = 2x^2+3x-5$. We already found that $\frac{\mathrm{d}f}{\mathrm{d}x}(x) = 4x+3$. Taking the derivative of $\frac{\mathrm{d}f}{\mathrm{d}x}$, we get that

$$\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}(x) = 4.$$

The fact that the second derivative of $f(x)$ is positive everywhere will tell us quite a bit about the shape of the graph of $f(x)$. We will learn more about the information the second derivative gives us in the next lecture.

Finally, it should be noted that very often we can find the derivative of the second derivative of a function, which we call the third derivative, and keep taking derivatives over and over again. The ith derivative of $f(x)$, the function we get by taking the derivative i times, is written using the notation $\frac{\mathrm{d}^i f}{\mathrm{d}x^i}$. The second derivative of $f(x)$ and all of the derivatives after it are called the higher derivatives of $f(x)$. Of them, the second derivative is by far the most important, but we will find uses for other higher derivatives as the terms moves along.