

Directional Derivatives and the Gradient Vector

Lecture 24

February 26, 2007

Fact

Recall:

$$f_x(x_0, y_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$
$$f_y(x_0, y_0) = \lim_{h \rightarrow 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}.$$

The Directional Derivative

Definition

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- The **directional derivative** of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b \rangle$ is

$$D_{\mathbf{u}}f(x_0, y_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

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- If $\mathbf{u} = \mathbf{i} = \langle 1, 0 \rangle$, then $D_{\mathbf{i}}f = f_x$, and if $\mathbf{u} = \mathbf{j} = \langle 0, 1 \rangle$, then $D_{\mathbf{j}} = f_y$.

Theorem

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- If f is a differentiable function of x and y , then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b \rangle$ and

$$D_{\mathbf{u}}f(x, y) = f_x(x, y)a + f_y(x, y)b.$$

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- If the unit vector \mathbf{u} makes an angle θ with the positive x -axis, then we can write $\mathbf{u} = \langle \cos \theta, \sin \theta \rangle$ and

$$D_{\mathbf{u}}f(x, y) = f_x(x, y) \cos \theta + f_y(x, y) \sin \theta.$$

Examples

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- Find the directional derivative of

$$f(x, y) = x^3 - 3xy + 4y^2$$

at the point $(1, 2)$ in the direction $\theta = \pi/6$.

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- Find the directional derivative of $f(x, y) = xe^y + \cos(xy)$ at the point $(2, 0)$ in the direction of $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$.

Definition

- If f is a function of two variables x and y , then the **gradient** of f is the vector function ∇f defined by

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j}.$$

Example

- Find the gradient of $f(x, y) = \sin x + e^{xy}$ at $(0, 1)$.

Fact

- *The equation of the directional derivative becomes:*

$$D_{\mathbf{u}}f(x, y) = \nabla f(x, y) \cdot \mathbf{u}.$$

Example

Find the directional derivative of the function $f(x, y) = x^2y^3 - 4y$ at the point $(2, -1)$ in the direction of the vector $\mathbf{v} = 2\mathbf{i} + 5\mathbf{j}$.

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- If $w = f(x, y, z)$ is a function of three variables, the **directional derivative** of f at (x_0, y_0, z_0) in the direction of the unit vector $\langle a, b, c \rangle$ is

$$D_u f(x_0, y_0, z_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + ha, y_0 + hb, z_0 + hc) - f(x_0, y_0, z_0)}{h}$$

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- Then

$$D_u f(x, y, z) = f_x(x, y, z)a + f_y(x, y, z)b + f_z(x, y, z)c.$$

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$$\nabla f = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k}$$

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$$\nabla f = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k}$$

- The formula for the directional derivative become

$$D_{\mathbf{u}} f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}.$$

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Consider the function $f(x, y, z) = xy^2 + yz^3 + xy^2$.

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- Find the gradient of f at the point $(5, 4, -1)$.

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- Find the gradient of f .
- Find the gradient of f at the point $(5, 4, -1)$.
- Find the rate of change of the function f at the point $(4, 5, -1)$ in the direction $\mathbf{u} = \langle 2/\sqrt{20}, -3/\sqrt{20}, -3/\sqrt{20} \rangle$.